Stochastic differential games for fully coupled FBSDEs with jumps *

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Abstract. This paper is concerned with stochastic differential games (SDGs) defined through fully coupled forward-backward stochastic differential equations (FBSDEs) which are governed by Brownian motion and Poisson random measure. For SDGs, the upper and the lower value functions are defined by the controlled fully coupled FBSDEs with jumps. Using a new transformation introduced in [6], we prove that the upper and the lower value functions are deterministic. Then, after establishing the dynamic programming principle for the upper and the lower value functions of this SDGs, we prove that the upper and the lower value functions are the viscosity solutions to the associated upper and the lower Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, respectively. Furthermore, for a special case (when $\sigma,\ h$ do not depend on $y,\ z,\ k$), under the Isaacs' condition, we get the existence of the value of the game.

Keyword. Fully coupled FBSDEs with jumps, stochastic differential game, Hamilton-Jacobi-Bellman-Isaacs equation, value function, stochastic backward semigroup, dynamic programming principle, viscosity solution

1 Introduction

General nonlinear backward stochastic differential equations (BSDEs, for short) in the framework of Brownian motion were first introduced by Pardoux, Peng in [24]. They got the uniqueness and the existence theorem for nonlinear BSDEs under Lipschitz condition. Since then, the theory of BSDEs has been studied widely, namely in stochastic control (see Peng [28]), finance (see El Karoui, Peng and Quenez [10]), and the theory of partial differential equations (PDEs, for short) (see Pardoux, Peng [25], Peng [29], etc). Related tightly with the BSDE theory, the theory of fully coupled forward-backward stochastic differential equations (FBSDEs, for short) has shown a dynamic development. Fully coupled FBSDEs driven by Brownian motion are encountered in the optimization problem when applying stochastic maximum principle. Also, in finance, fully coupled FBSDEs are often used when considering problems with the large investors, see [8, 22]. On one hand, for the existence and uniqueness of solutions of fully coupled FBSDEs driven by Brownian motion, the reader can refer to Antonelli [1], Delarue [9], Hu, Peng [15], Ma, Protter and Yong [20], Ma, Wu, Zhang and Zhang [21], Ma, Yong [22], Pardoux, Tang [26], Peng, Wu [31], Yong [37, 38], Zhang [39], etc. Pardoux, Tang [26] associated fully coupled FBSDEs driven by Brownian motion (without controls and σ doesn't depend on z) with quasilinear parabolic PDEs, and gave an existence result for viscosity solution. Wu, Yu [35, 36] proved the existence of a quasilinear PDEs with the help of fully coupled FBSDEs driven by Brownian motion when σ depends on z, but their stochastic systems are without controls. Recently, Li, Wei

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[18] studied the stochastic optimal control problems of fully coupled FBSDEs driven by Brownian motion in two cases: (i) the diffusion coefficient σ depends on z, i.e., depends on the second component of the solution (Y, Z) of the BSDE and does not depend on the control u; (ii) σ depends on the control u, but does not depend on z. They also proved some new estimates for fully coupled FBSDEs on small time interval which were used for the proof of the viscosity solution. Li [16] studied the general case, that is, σ depends on z and the control u at the same time.

BSDEs with Poisson random measure were first discussed by Tang, Li [32]. Later, Barles, Buckdahn and Pardoux [2] proved that the solutions of the BSDEs driven by a Brownian motion and a Poisson random measure provide the viscosity solutions for the associated system of parabolic integral-partial differential equations. In [17], using Peng's BSDE approach, Li, Peng studied the stochastic control theory for BSDE with jumps.

On the other hand, as concerns stochastic differential games (SDGs, for short), two-player zero-sum SDGs of the type of strategy against control, they were first studied by Fleming, Souganidis [11] in 1989. In their paper under the Isaacs' condition the lower and the upper value functions of the game coincide, satisfy the dynamic programming principle (DPP, for short), and they are the unique viscosity solution of the associated Bellman-Isaacs equation. Since then there are a lot of works about this topic, such as, Buckdahn, Li [5], they gave a more general but also a more direct approach than that in [11], and also Buckdahn, Cardaliaguet and Rainer [4], Buckdahn, Li and Hu [6], Hou, Tang [14] and so on.

BSDEs methods, were introduced originally by Peng [27, 29, 30] for the stochastic control theory. Since then BSDE methods have been extended to the theory of SDGs. Hamadene, Lepeltier [12] and Hamadene, Lepeltier and Peng [13] studied games with a dynamics whose diffusion coefficient is strictly elliptic and does not depend on controls. Buckdahn, Li [5] studied two-player zero-sum SDGs with the help of decoupled FBSDEs driven by Brownian motion. They introduced the method of Girsanov transformation which turned out to be a straightforward way to prove that the upper and the lower value functions of the game are deterministic. However, this method can't be applied to SDGs with jumps. Buckdahn, Li and Hu [6] introduced a new type of measure-preserving and invertible transformation on the Wiener-Poisson space to prove that the upper and the lower value functions for two-player zero-sum SDGs with jumps are deterministic. And the proof that they are deterministic does not depend on the BSDE methods so that the new method can be used for the standard stochastic control problems with jumps. In [19], Li, Wei studied some useful estimates for fully coupled FBSDEs with jumps under the monotonic condition. Moreover, under Lipschitz condition and linear growth condition, they established the existence and the uniqueness of the solution and prove L^p-estimates on a small time interval, which play an important role in the study of the existence of the viscosity solution for the corresponding second order integral-partial differential equation of Isaacs' type over an arbitrary time interval.

Inspired by the control problems in Li [16], Li, Wei [18], as well as Buckdahn, Li and Hu [6], we will investigate SDGs defined through fully coupled FBSDEs driven by Brownian motion and Poisson random measure, where $\sigma,\ h$ depend on z and the controls $u,\ v$ at the same time. For the fully coupled FBSDEs with jumps, under the monotonicity assumptions Wu [33] obtained the existence and the uniqueness of the solution. Later, Wu [34] proved the existence and the uniqueness of the solution as well as a comparison theorem for fully coupled FBSDEs with jumps over a stochastic interval. Similarly to [16, 18], the second order integral-partial differential equations of Isaacs' type are also combined with the algebraic equations. Therefore, we still need the representation theorem for the related algebraic equation which is got in [18].

Precisely, in this paper, the cost functional (interpreted as a payoff for Player I and as a cost for Player II) of our SDGs is introduced by the following fully coupled FBSDE driven by Brownian motion and Poisson random measure:

$$\left\{\begin{array}{rcl} dX_s^{t,x;u,v} &=& b(s,\Pi_s^{t,x;u,v},u_s,v_s)ds+\sigma(s,\Pi_s^{t,x;u,v},u_s,v_s)dB_s+\int_E h(s,\Pi_{s-}^{t,x;u,v},u_s,v_s)\tilde{\mu}(ds,de),\\ dY_s^{t,x;u,v} &=& -f(s,\Pi_s^{t,x;u,v},\int_E K_s^{t,x;u,v}(e)l(e)\lambda(de),u_s,v_s)ds+Z_s^{t,x;u,v}dB_s+\int_E K_s^{t,x;u,v}(e)\tilde{\mu}(ds,de),\\ X_t^{t,x;u,v} &=& \zeta,\\ Y_T^{t,x;u,v} &=& \Phi(X_T^{t,x;u,v}), \end{array}\right. \tag{1.1}$$

where $s\in[t,T]$, $\Pi_s^{t,x;u,v}=(X_s^{t,x;u,v},Y_s^{t,x;u,v},Z_s^{t,x;u,v})$, $\Pi_{s-}^{t,x;u,v}=(X_{s-}^{t,x;u,v},Y_{s-}^{t,x;u,v},Z_s^{t,x;u,v})$, $T>0$ is an arbitrarily fixed finite time horizon, and the admissible controls $u=(u_s)_{s\in[t,T]}\in\mathcal{U}_{t,T}$, $v=(v_s)_{s\in[t,T]}\in\mathcal{V}_{t,T}$

are predictable and take their values in a compact metric space U and V, respectively. Under our assumptions (see Section 2), the equation (1.1) has a unique solution $(X_s^{t,x;u,v}, Y_s^{t,x;u,v}, Z_s^{t,x;u,v}, K_s^{t,x;u,v})_{s\in[t,T]}$ and the cost functional is defined by

$$J(t,x;u,v) := Y_t^{t,x;u,v}.$$

We define the lower value function and the upper value function of our SDG, respectively, as follows

$$W(t,x) := \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u)),$$

$$U(t,x) := \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} \operatorname*{essinf}_{v\in\mathcal{V}_{t,T}} J(t,x;\alpha(v),v),$$

where $\mathcal{A}_{t,T},\ \mathcal{B}_{t,T}$ are the sets of nonanticipative strategies of Player I and Player II, respectively (see Definition 3.2 in Section 3). The objective of our paper is to investigate the lower and the upper value functions. The main results of the paper state that W and U are deterministic (Proposition 3.1), satisfy the DPP (Theorem 3.1), and are continuous viscosity solutions of the associated Bellman-Isaacs equations (Theorem 4.1). In our approach, we will use in a crucial manner the results for fully coupled FBSDEs with jumps on the small time interval obtained by Li, Wei [19].

Our paper is organized as follows. In Section 2, we present some preliminaries for BSDEs with jumps and fully coupled FBSDEs with jumps, which will be used later. The setting of our SDGs is introduced in Section 3. We also show that the lower and upper value functions (3.5), (3.6) are deterministic functions, Lipschitz in x (Lemma 3.3) and $\frac{1}{2}$-Hölder continuous in t (Theorem 3.2). Moreover, they satisfy the DPP (Theorem 3.1). In Section 4, by using the DPP, we prove that W and U are the viscosity solutions of the associated integral-differential Bellman-Isaacs equation. Furthermore, Section 5 presents the uniqueness of viscosity solution for the case when $\sigma,\ h$ does not depend on $y,\ z,\ k$. This shows that, under Isaacs' condition this game has a value. Finally, in Appendix we give the proof of the DPP.

2 Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a probability space which is the completed product of the Wiener space $(\Omega_1,\mathcal{F}_1,P_1)$ and the Poisson space $(\Omega_2,\mathcal{F}_2,P_2)$.

• $(\Omega_1,\mathcal{F}_1,P_1)$ is a classical Wiener space, where $\Omega_1 = C_0(\mathbb{R};\mathbb{R}^d)$ is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$ with value 0 in time 0, $\mathcal{F}_1$ is the completed Borel σ-algebra over Ω_1, and P_1 is the Wiener measure such that $B_s(\omega)=\omega_s,\ s\in\mathbb{R}_+,\ \omega\in\Omega_1$, and $B_{-s}(\omega)=\omega(-s),\ s\in\mathbb{R}_+,\ \omega\in\Omega_1$, are two independent d-dimensional Brownian motions. The natural filtration $\{\mathcal{F}_s^B, s\ge 0\}$ is generated by $\{B_s\}_{s\ge0}$ and augmented by all P_1-null sets, i.e.,

$$\mathcal{F}_s^B = \sigma\{B_r, r\in(-\infty,s]\}\vee\mathcal{N}_{P_1},\ s\ge 0.$$

• $(\Omega_2,\mathcal{F}_2,P_2)$ is a Poisson space. $p: D_p\subset\mathbb{R}\to E$ is a point function, where D_p is a countable subset of the real line $\mathbb{R}$, $E=\mathbb{R}^l\backslash\{0\}$ is equipped with its Borel σ-field $\mathcal{B}(E)$. We introduce the counting measure $\mu(p,dtde)$ on $\mathbb{R}\times E$ as follows:

$$\mu(p,(s,t]\times\Delta) = \sharp\{r\in D_p\cap(s,t]: p(r)\in\Delta\},\ \Delta\in\mathcal{B}(E),\ s,t\in\mathbb{R},\ s<t,$$

where $\sharp$ denotes the cardinal number of the set. We identify the point function p with $\mu(p,\cdot)$. Let Ω_2 be the set of all point functions p on E, and $\mathcal{F}_2$ be the smallest σ-field on Ω_2. The coordinate mappings $p\to\mu(p,(s,t]\times\Delta),\ s,t\in\mathbb{R},\ s<t,\ \Delta\in\mathcal{B}(E)$ are measurable with respect to $\mathcal{F}_2$. On the measurable space $(\Omega_2,\mathcal{F}_2)$ we consider the probability measure P_2 such that the canonical coordinate measure $\mu(p,dtde)$ becomes a Poisson random measure with the compensator $\hat\mu(dtde)=dt\lambda(de)$ and the process $\{\tilde\mu((s,t]\times A) = (\mu-\hat\mu)((s,t]\times A)\}_{s\le t}$ is a martingale, for any $A\in\mathcal{B}(E)$ satisfying $\lambda(A)<\infty$. Here λ is supposed to be a σ-finite measure on $(E,\mathcal{B}(E))$ with $\int_E(1\wedge|e|^2)\lambda(de)<\infty$. The filtration $\{\mathcal{F}_t^\mu\}_{t\ge0}$ generated by the coordinate measure μ is introduced by setting:

$$\dot{\mathcal{F}}_t^\mu = \sigma\{\mu((s,r]\times\Delta): -\infty<s\le r\le t, \Delta\in\mathcal{B}(E)\},\ t\ge 0,$$

and taking the right-limits $\mathcal{F}_t^\mu = (\bigcap_{s>t} \dot{\mathcal{F}}_s^\mu) \vee \mathcal{N}_{P_2},\ t \geq 0$, augmented by all the P_2-null sets. At last, we set $(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$, where $\mathcal{F}$ is completed with respect to P, and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by

$$\mathcal{F}_t := \mathcal{F}_t^B \otimes \mathcal{F}_t^\mu,\ t \geq 0,\ \text{augmented by all } P\text{-null sets.}$$

For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. Fix $T > 0$, we introduce the following spaces of processes which will be used later.

- $\mathcal{M}^2(t, T; \mathbb{R}^d) := \Big\{\varphi \mid \varphi : \Omega \times [t, T] \to \mathbb{R}^d$ is an $\mathbb{F}$-predictable process : $\parallel \varphi \parallel^2 = E[\int_t^T |\varphi_s|^2 ds] < +\infty\Big\}$;
- $\mathcal{S}^2(t, T; \mathbb{R}) := \Big\{\psi \mid \psi : \Omega \times [t, T] \to \mathbb{R}$ is an $\mathbb{F}$-adapted càdlàg process : $E[\sup_{t\leq s\leq T} |\psi_s|^2] < +\infty\Big\}$;
- $\mathcal{K}_\lambda^2(t, T; \mathbb{R}^n) := \Big\{K \mid K : \Omega \times [t, T] \times E \to \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{B}(E) -$ measurable :
  $\parallel K \parallel^2 = E[\int_t^T \int_E |K_s(e)|^2 \lambda(de) ds] < +\infty\Big\}$,[1]

where $t \in [0, T]$.

2.1 BSDEs with jumps

Let us consider the following BSDE with jumps:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, K_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E K_s(e) \tilde{\mu}(ds, de),\ t \in [0, T]. \tag{2.1}$$

where $T > 0$ is an arbitrary time horizon, and the coefficient $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \to \mathbb{R}$ is $\mathcal{P}$-measurable for each $(y, z, k) \in \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$ and satisfies:

(**H2.1**) (i) There exists a constant $C \geq 0$ such that, P-a.s., for all $t \in [0, T],\ y_1, y_2 \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d,\ k_1, k_2 \in L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$,

$$|g(t, y_1, z_1, k_1) - g(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \parallel k_1 - k_2 \parallel);$$

(ii) $E[(\int_0^T |g(s, 0, 0, 0)| ds)^2] < +\infty$.

Let us recall some well-known results.

Lemma 2.1. *Under the assumption* (**H2.1**)*, for any random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, the BSDE with jumps (2.1) has a unique adapted solution*

$$(Y_t, Z_t, K_t)_{t\in[0,T]} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{M}^2(0, T; \mathbb{R}^d) \times \mathcal{K}_\lambda^2(0, T; \mathbb{R}).$$

Lemma 2.2. *(Comparison Theorem) Let $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ measurable and satisfy*

(i) *there exists a constant $C \geq 0$ such that, P-a.s., for all $t \in [0, T],\ y_1, y_2 \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d,\ k_1, k_2 \in \mathbb{R}$,*

$$|a(t, y_1, z_1, k_1) - a(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |k_1 - k_2|).$$

(ii) $a(\cdot, 0, 0, 0) \in \mathcal{M}^2(0, T; \mathbb{R})$.

(iii) *$k \to a(t, y, z, k)$ is non-decreasing, for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$.*

[1] $\mathcal{P}$ denotes the σ-field of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$.

Furthermore, let $l:\Omega\times[0,T]\times E\rightarrow\mathbb{R}$ *be* $\mathcal{P}\otimes\mathcal{B}(E)$ *measurable and satisfy*

$$0\leq l_t(e)\leq C(1\wedge|e|),\quad e\in E.$$

We set

$$g(\omega,t,y,z,\varphi)=a(\omega,t,y,z,\int_E\varphi(e)l_t(\omega,e)\lambda(de)),$$

for $(\omega,t,y,z,\varphi)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R})$.

Let $\xi,\ \xi'\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$ *and* g' *satisfies* (**H2.1**).

We denote by (Y,Z,K) *(resp.,* (Y',Z',K')*) the unique solution of equation (2.1) with the data* (ξ,g) *(resp.,* (ξ',g')*). If*

(iv) $\xi\geq\xi',$ *a.s.*;

(v) $g(t,y,z,k)\geq g'(t,y,z,k),$ *a.s., a.e., for any* $(y,z,k)\in\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R})$,

then, we have: $Y_t\geq Y'_t,$ *a.s., for all* $t\in[0,T]$*. And if, in addition, we also assume that* $P(\xi>\xi')>0$*, then* $P(Y_t>Y'_t)>0,\ 0\leq t\leq T$*, and in particular,* $Y_0>Y'_0$*.*

Using the notation introduced in Lemma 2.2, we suppose that, for some $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R})\rightarrow\mathbb{R}$ satisfying (**H2.1**), and for $i\in\{1,2\}$, the drivers g_i are of the form

$$g_i(s,y_s^i,z_s^i,k_s^i)=g(s,y_s^i,z_s^i,k_s^i)+\varphi_i(s),\ dsdP\text{-a.e.}$$

Lemma 2.3. *The difference of the solutions* (Y^1,Z^1,K^1) *and* (Y^2,Z^2,K^2) *of BSDE (2.1) with the data* (ξ_1,g_1) *and* (ξ_2,g_2)*, respectively, satisfies the following estimate:*

$$\begin{array}{l}|Y_t^1-Y_t^2|^2+\frac{1}{2}E[\int_t^T e^{\beta(s-t)}(|Y_s^1-Y_s^2|^2+|Z_s^1-Z_s^2|^2)ds|\mathcal{F}_t]\\ +\frac{1}{2}E[\int_t^T\int_E e^{\beta(s-t)}|K_s^1(e)-K_s^2(e)|^2\lambda(de)ds|\mathcal{F}_t]\\ \leq E[e^{\beta(T-t)}|\xi_1-\xi_2|^2|\mathcal{F}_t]+E[\int_t^T e^{\beta(s-t)}|\varphi_1(s)-\varphi_2(s)|^2ds|\mathcal{F}_t],\ P\text{-a.s., for all } 0\leq t\leq T,\end{array}$$

where $\beta\geq 2+2C+4C^2$.

The reader may refer to Barles, Buckdahn and Pardoux [2] for the proof.

2.2 Fully coupled FBSDEs with jumps

Now we consider the following fully coupled FBSDE with jumps associated with $(b,\sigma,h,f,\zeta,\Phi)$ on the time interval $[t,T]$, where $t\in[0,T]$:

$$\left\{\begin{array}{rcl}dX_s&=&b(s,X_s,Y_s,Z_s,K_s)ds+\sigma(s,X_s,Y_s,Z_s,K_s)dB_s+\int_E h(s,X_{s-},Y_{s-},Z_s,K_s(e),e)\tilde{\mu}(ds,de),\\ dY_s&=&-f(s,X_s,Y_s,Z_s,\int_E K_s(e)l(e)\lambda(de))ds+Z_sdB_s+\int_E K_s(e)\tilde{\mu}(ds,de),\quad s\in[t,T],\\ X_t&=&\zeta,\\ Y_T&=&\Phi(X_T),\end{array}\right.\tag{2.2}$$

where (X,Y,Z,K) takes its values in $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m$, and

$$\begin{array}{l}b:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)\longrightarrow\mathbb{R}^n,\\ \sigma:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)\longrightarrow\mathbb{R}^{n\times d},\\ h:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times E\longrightarrow\mathbb{R}^n,\\ f:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\longrightarrow\mathbb{R}^m,\end{array}$$

$l:E\longrightarrow\mathbb{R}$ and $\Phi:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^m$ satisfy

(**H2.2**) (i) $b,\ \sigma,\ f$ are uniformly Lipschitz with respect to (x,y,z,k), and there exists $\rho:E\rightarrow\mathbb{R}^+$ with $\int_E\rho^2(e)\lambda(de)<+\infty$ such that, for any $t\in[0,T],\ x,\bar{x}\in\mathbb{R}^n,\ y,\bar{y}\in\mathbb{R}^m,\ z,\bar{z}\in\mathbb{R}^{m\times d},\ k,\bar{k}\in\mathbb{R}^m$ and $e\in E$,

$$|h(t,x,y,z,k,e)-h(t,\bar{x},\bar{y},\bar{z},\bar{k},e)|\leq\rho(e)(|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|+|k-\bar{k}|);$$

(ii) $k \to f(t,x,y,z,k)$ is non-decreasing, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$;

(iii) there exists a constant $C > 0$ such that

$$0 \le l(e) \le C(1 \wedge |e|),\ x \in \mathbb{R}^n,\ e \in E;$$

(iv) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^n$;

(v) for every $(x,y,z,k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$, $b,\ \sigma,\ h,\ f$ are $\mathbb{F}$-progressively measurable and

$$\begin{array}{l} E\int_0^T |b(s,0,0,0,0)|^2 ds + E\int_0^T |f(s,0,0,0,0)|^2 ds + E\int_0^T |\sigma(s,0,0,0,0)|^2 ds \\ +E\int_0^T \int_E |h(s,0,0,0,0,e)|^2 \lambda(de) ds < \infty. \end{array}$$

Let

$$g(s,x,y,z,k) := f(s,x,y,z,\int_E k(e)l(e)\lambda(de)),$$

$(s,x,y,z,k) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$.

In this paper we use the usual inner product and the Euclidean norm in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively. Given an $m \times n$ full-rank matrix G, we define:

$$\pi = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A(t,\pi,k) = \begin{pmatrix} -G^T g \\ Gb \\ G\sigma \end{pmatrix}(t,\pi,k),$$

where G^T is the transposed matrix of G.

We assume the following monotonicity conditions:

(H2.3) (i) $$\begin{array}{l} \langle A(t,\pi,k) - A(t,\bar\pi,\bar k), \pi - \bar\pi\rangle + \int_E \langle G\widehat{h}(e), \widehat{k}(e)\rangle \lambda(de) \\ \le -\beta_1 |G\widehat{x}|^2 - \beta_2(|G^T\widehat{y}|^2 + |G^T\widehat{z}|^2) - \beta_3 \int_E |G^T\widehat{k}(e)|^2 \lambda(de), \end{array}$$

(ii) $\langle \Phi(x) - \Phi(\bar x), G(x - \bar x)\rangle \ge \mu_1 |G\widehat{x}|^2$, $\forall \pi = (x,y,z)$, $\bar\pi = (\bar x, \bar y, \bar z)$, $\widehat{x} = x - \bar x$, $\widehat{y} = y - \bar y$, $\widehat{z} = z - \bar z$, $\widehat{k} = k - \bar k$, $\widehat{h}(e) = h(t,\pi,k,e) - h(t,\bar\pi,\bar k,e)$,

where $\beta_1,\ \beta_2,\ \beta_3,\ \mu_1$ are nonnegative constants with $\beta_1+\beta_2 > 0$, $\beta_1+\beta_3 > 0$, $\beta_2+\mu_1 > 0$, $\beta_3+\mu_1 > 0$. Moreover, we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$, $\beta_3 > 0$), when $m > n$ (resp., $m < n$).

Remark 2.1. **(H2.3)-(ii)'** *A consequence of* (**H2.3**) (ii) *is the weaker condition:* $\langle \Phi(x) - \Phi(\bar x), G(x - \bar x)\rangle \ge 0$*, for all* $x,\ \bar x \in \mathbb{R}^n$*.*

When $\Phi(x) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$, (**H2.3**)-(i) can be weaken as follows:

(H2.4) $\langle A(t,\pi,k) - A(t,\bar\pi,\bar k), \pi - \bar\pi\rangle + \int_E \langle G\widehat{h}(e), \widehat{k}(e)\rangle \lambda(de) \le -\beta_1 |G\widehat{x}|^2 - \beta_2 |G^T\widehat{y}|^2$,

where $\beta_1,\ \beta_2$ are nonnegative constants with $\beta_1 + \beta_2 > 0$. Moreover, we have $\beta_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

Lemma 2.4. *Under the assumptions* (**H2.2**) *and* (**H2.3**)*, for any* $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$*, FBSDE (2.2) has a unique adapted solution* $(X_s, Y_s, Z_s, K_s)_{s\in[t,T]} \in \mathcal{S}^2(t,T;\mathbb{R}^n) \times \mathcal{S}^2(t,T;\mathbb{R}^m) \times \mathcal{M}^2(t,T;\mathbb{R}^{m\times d}) \times \mathcal{K}^2_\lambda(t,T;\mathbb{R}^m)$.

Lemma 2.5. *Under the assumptions* (**H2.3**)-(ii)' *and* (**H2.4**)*, for any* $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ *and the terminal condition* $\Phi(x) = \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$*, FBSDE (2.2) has a unique adapted solution* $(X_s, Y_s, Z_s, K_s)_{s\in[t,T]} \in \mathcal{S}^2(t,T;\mathbb{R}^n) \times \mathcal{S}^2(t,T;\mathbb{R}^m) \times \mathcal{M}^2(t,T;\mathbb{R}^{m\times d}) \times \mathcal{K}^2_\lambda(t,T;\mathbb{R}^m)$.

For the proof, the reader can refer to Wu [33, 34].

Now we recall the comparison theorem for fully coupled FBSDEs with jumps.

Lemma 2.6. *(Comparison Theorem) Let* $m = 1$ *and assume that* $(b, \sigma, h, f^i, a^i, \Phi^i)$*, for* $i = 1, 2$*, satisfy* (**H2.2**), (**H2.3**)*, where* $b,\ \sigma,\ h$ *do not depend on* k*,* a_i *is the initial state. Let* $(x^i_s, y^i_s, z^i_s, k^i_s)_{t\le s\le T}$ *be the solution of FBSDE (2.2) associated with* $(b, \sigma, h, f^i, a^i, \Phi^i)$ *on the time interval* $[t,T]$*. We assume* $a^1 \ge a^2$*,* $\Phi^1(x) \ge \Phi^2(x)$*,* $f^1(t,x,y,z,k) \ge f^2(t,x,y,z,k)$*, P-a.s., for all* $(x,y,z,k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$*, then,* $y^1_t \ge y^2_t$*, P-a.s.*

The above lemma can be found in [34].

3 A DPP for stochastic differential games for fully coupled FBSDEs with jumps

In this section, we consider stochastic differential games for fully coupled FBSDEs with jumps. First we introduce the background of stochastic differential games. Suppose that the control state spaces $U,\ V$ are compact metric spaces. By $\mathcal{U}$ (resp., $\mathcal{V}$) we denote the admissible control set of all U (resp., V)-valued $\mathcal{F}_t$-predictable processes for the first (resp., second) player. If $u \in \mathcal{U}$ (resp., $v \in \mathcal{V}$), we call u (resp., v) an admissible control.

Let us give the following deterministic measurable functions

$$\begin{array}{ll} b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \times V \longrightarrow \mathbb{R}, & \sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \times V \longrightarrow \mathbb{R}^d, \\ h : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \times V \times E \longrightarrow \mathbb{R}, & f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \times V \longrightarrow \mathbb{R}, \end{array}$$

and $l : E \longrightarrow \mathbb{R},\ \Phi : \mathbb{R} \longrightarrow \mathbb{R}$, which are continuous in (t, u, v), and satisfy the assumptions $(\mathbf{H2.2}),\ (\mathbf{H2.3})$, uniformly in $u \in U,\ v \in V$.

For given admissible controls $u(\cdot) \in \mathcal{U},\ v(\cdot) \in \mathcal{V}$ and the initial data $(t, \zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R})$, we consider the following fully coupled forward-backward stochastic system

$$\left\{\begin{array}{rcl} dX_s^{t,\zeta;u,v} &=& b(s, \Pi_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, \Pi_s^{t,\zeta;u,v}, u_s, v_s)dB_s + \int_E h(s, \Pi_{s-}^{t,\zeta;u,v}, u_s, v_s, e)\tilde{\mu}(ds, de), \\ dY_s^{t,\zeta;u,v} &=& -f(s, \Pi_s^{t,\zeta;u,v}, \int_E K_s^{t,\zeta;u,v}(e)l(e)\lambda(de), u_s, v_s)ds + Z_s^{t,\zeta;u,v}dB_s + \int_E K_s^{t,\zeta;u,v}(e)\tilde{\mu}(ds, de), \\ X_t^{t,\zeta;u,v} &=& \zeta, \\ Y_T^{t,\zeta;u,v} &=& \Phi(X_T^{t,\zeta;u,v}), \end{array}\right. \tag{3.1}$$

where $s \in [t,T],\ \Pi_s^{t,\zeta;u,v} = (X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v})$ and $\Pi_{s-}^{t,\zeta;u,v} = (X_{s-}^{t,\zeta;u,v}, Y_{s-}^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v})$.

Therefore, for any $u(\cdot) \in \mathcal{U},\ v(\cdot) \in \mathcal{V}$, from Lemma 2.4, we have that FBSDE (3.1) has a unique solution.

Remark 3.1. *Due to the restrictions coming from the comparison theorem (Lemma 2.6, Theorem 3.3 in [19]) which will be used in Section 4, we emphasize that the coefficients $b,\ \sigma,\ h$ do not depend on the variable k.*

Remark 3.2. *Under our assumptions, it is obvious that b, σ, h, f, Φ have linear growth in $(\pi, k) = (x, y, z, k)$, i.e.,*

$$\begin{array}{l} |b(t,\pi,u,v)| + |\sigma(t,\pi,u,v)| + |f(t,\pi,k,u,v)| + |\Phi(x)| \le C(1 + |x| + |y| + |z| + |k|), \\ |h(t,\pi,u,v,e)| \le \rho(e)(1 + |x| + |y| + |z|), \end{array}$$

for $(t, x, y, z, k, u, v) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \times V$.

From Proposition 3.1 in [19], for our FBSDE with jumps (3.1), it is easy to check that, there exists $C \in \mathbb{R}^+$ such that, for any $t \in [0,T],\ \zeta,\ \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}),\ u(\cdot) \in \mathcal{U},\ v(\cdot) \in \mathcal{V}$, we have, P-a.s.:

$$\begin{array}{ll} \text{(i)} & |Y_t^{t,\zeta;u,v}| \le C(1 + |\zeta|); \\ \text{(ii)} & |Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \le C|\zeta - \zeta'|. \end{array} \tag{3.2}$$

Now, we introduce the subspaces of admissible controls and the definition of admissible strategies, which are similar to [5, 6, 11].

Definition 3.1. *An admissible control process $u = (u_r)_{r\in[t,s]}$ (resp., $v = (v_r)_{r\in[t,s]}$) for Player I (resp., II) on $[t, s]$ is an $\mathcal{F}_r$-predictable, U (resp., V)-valued process. The set of all admissible controls for Player I (resp., II) on $[t, s]$ is denoted by $\mathcal{U}_{t,s}$ (resp., $\mathcal{V}_{t,s}$). If $P\{u \equiv \bar{u},\ a.e.,\ in\ [t,s]\} = 1$, we will identify both processes u and $\bar{u}$ in $\mathcal{U}_{t,s}$. Similarly we interpret $v \equiv \bar{v}$ on $[t, s]$ in $\mathcal{V}_{t,s}$.*

Definition 3.2. *A nonanticipative strategy for Player I on $[t, s]$ $(t < s \le T)$ is a mapping $\alpha :\ \mathcal{V}_{t,s} \to \mathcal{U}_{t,s}$ such that, for any $\mathcal{F}_r$-stopping time $S : \Omega \to [t, s]$ and any $v_1, v_2 \in \mathcal{V}_{t,s}$, with $v_1 \equiv v_2$ on $[[t, S]]$, it holds that $\alpha(v_1) \equiv \alpha(v_2)$ on $[[t, S]]$. Nonanticipative strategies for Player II on $[t, s]$, $\beta :\ \mathcal{U}_{t,s} \to \mathcal{V}_{t,s}$, are defined similarly. The set of all nonanticipative strategies $\alpha :\ \mathcal{V}_{t,s} \to \mathcal{U}_{t,s}$ for Player I on $[t, s]$ is denoted by $\mathcal{A}_{t,s}$. The set of all nonanticipative strategies $\beta :\ \mathcal{U}_{t,s} \to \mathcal{V}_{t,s}$ for Player II on $[t, s]$ is denoted by $\mathcal{B}_{t,s}$. (Recall that $[[t, S]] = \{(r, \omega) \in [0,T] \times \Omega, t \le r \le S(\omega).$)*

For given processes $u(\cdot) \in \mathcal{U}_{t,T},\ v(\cdot) \in \mathcal{V}_{t,T}$, the cost functional is defined as follows:

$$J(t,x;u,v) := Y_t^{t,x;u,v}, \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{3.3}$$

where the process $Y^{t,x;u,v}$ is defined by FBSDE (3.1).

From Theorem 3.1 in [19] we have, for any $t \in [0,T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R})$,

$$J(t,\zeta;u,v) := Y_t^{t,\zeta;u,v}, \text{ P-a.s.} \tag{3.4}$$

For $\zeta = x \in \mathbb{R}$, we define the lower value function of our stochastic differential games

$$W(t,x) := \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u)), \tag{3.5}$$

and its upper value function

$$U(t,x) := \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{essinf}_{v \in \mathcal{V}_{t,T}} J(t,x;\alpha(v),v). \tag{3.6}$$

Remark 3.3. *Thanks to the assumptions* (**H2.2**), (**H2.3**)*, the lower value function $W(t,x)$ and the upper value function $U(t,x)$ are well defined, and they are bounded $\mathcal{F}_t$-measurable random variables. But they even turn out to be deterministic.*

Next we will prove that $W,\ U$ are deterministic. The method of Girsanov transformation for fully coupled FBSDEs without jumps (see [5, 18]) does not apply to the case with jumps now. So we use a new transformation method introduced by Buckdahn, Li and Hu [6] to complete the proof that $W,\ U$ are deterministic. Next we only give the proof for W, that for U is similar.

Proposition 3.1. *For any $(t,x) \in [0,T] \times \mathbb{R}$, $W(t,x)$ is a deterministic function in the sense that $W(t,x) = E[W(t,x)]$, P-a.s.*

Combining the following both lemmas, we can complete the proof of this proposition.

Lemma 3.1. *Let $(t,x) \in [0,T] \times \mathbb{R}$ and $\tau : \Omega \to \Omega$ be an invertible $\mathcal{F} - \mathcal{F}$ measurable transformation such that*

(i) $\tau^{-1}(\mathcal{F}_t) \subset \mathcal{F}_t$ *and* $\tau(\mathcal{F}_t) \subset \mathcal{F}_t$*;*

(ii) $(B_s - B_t) \circ \tau = B_s - B_t,\ \mu((t,s] \times A) \circ \tau = \mu((t,s] \times A),\ s \in [t,T],\ A \in \mathcal{B}(E)$*;*

(iii) *the law $P \circ [\tau]^{-1}$ of τ is equivalent to the underlying probability measure P.*

Then, $W(t,x) \circ \tau = W(t,x)$, P-a.s.

Remark 3.4. *The assumptions* (i) *and* (ii) *of Lemma 3.1 imply that*

$$\tau^{-1}(\mathcal{F}_s) = \tau(\mathcal{F}_s) = \mathcal{F}_s,\ s \in [t,T].$$

Proof. We split the proof in the following steps:

(1). For any $u \in \mathcal{U}_{t,T},\ v \in \mathcal{V}_{t,T},\ J(t,x,u,v) \circ \tau = J(t,x,u(\tau),v(\tau))$, P-a.s.
In fact, applying the transformation τ to FBSDE (3.1) (with $\zeta = x$) and comparing the obtained equation with the FBSDE obtained from (3.1) by substituting the controlled processes $u(\tau),\ v(\tau)$ for u and v, we get from the uniqueness of the solution of (3.1) and the properties of the transformation τ that

$$\begin{array}{l}
X_s^{t,x;u,v}(\tau) = X_s^{t,x,u(\tau),v(\tau)}, \text{ for all } s \in [t,T], \text{ P-a.s.} \\
Y_s^{t,x;u,v}(\tau) = Y_s^{t,x,u(\tau),v(\tau)}, \text{ for all } s \in [t,T], \text{ P-a.s.} \\
Z_s^{t,x;u,v}(\tau) = Z_s^{t,x,u(\tau),v(\tau)}, \text{ dsdP-a.e. on } [t,T] \times \Omega, \\
K_s^{t,x;u,v}(\tau) = K_s^{t,x,u(\tau),v(\tau)}, \text{ ds}\lambda\text{(de)dP-a.e. on } [t,T] \times E \times \Omega.
\end{array}$$

Consequently, in particular, we have

$$J(t,x,u,v) \circ \tau = J(t,x,u(\tau),v(\tau)), \text{ P-a.s.}$$

(2). For $\beta \in \mathcal{B}_{t,T}$, let $\hat{\beta}(u) := \beta(u(\tau^{-1}))(\tau),\ u \in \mathcal{U}_{t,T}$. Then, $\hat{\beta} \in \mathcal{B}_{t,T}$.

Obviously, $\hat{\beta}$ maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, $\hat{\beta}$ is nonanticipative. Indeed, let $S : \Omega \to [t, T]$ be an $\mathbb{F}$-stopping time and $u_1, u_2 \in \mathcal{U}_{t,T}$ such that $u_1 \equiv u_2$ on $[[t, S]]$. Then, obviously, $u_1(\tau^{-1}) \equiv u_2(\tau^{-1})$ on $[[t, S(\tau^{-1})]]$ (notice that $S(\tau^{-1})$ is still an $\mathbb{F}$-stopping time. For this we use that the assumptions (i) and (ii) imply that $\tau(\mathcal{F}_s) := \{\tau(A), A \in \mathcal{F}_s\} = \mathcal{F}_s,\ s \in [t, T]$). Since $\beta \in \mathcal{B}_{t,T}$, we have $\beta(u_1(\tau^{-1})) \equiv \beta(u_2(\tau^{-1}))$ on $[[t, S(\tau^{-1})]]$. Therefore,

$$\hat{\beta}(u_1) = \beta(u_1(\tau^{-1}))(\tau) \equiv \beta(u_2(\tau^{-1}))(\tau) = \hat{\beta}(u_2), \text{ on } [[t, S]].$$

(3). For all $\beta \in \mathcal{B}_{t,T}$ we have

$$(\operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)))(\tau) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} (J(t, x; u, \beta(u))(\tau)), \text{ P-a.s.}$$

Indeed, let us use the notation $I(t, x, \beta) := \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)),\ \beta \in \mathcal{B}_{t,T}$, P-a.s. Then, $I(t, x, \beta)(\tau) \geq J(t, x; u, \beta(u))(\tau)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$. From the definition of essential supremum over a family of random variables, for any random variable ζ satisfying $\zeta \geq J(t, x; u, \beta(u))(\tau)$ and, hence, $\zeta(\tau^{-1}) \geq J(t, x; u, \beta(u))$, P-a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau^{-1}) \geq I(t, x, \beta)$, P-a.s., i.e., $\zeta \geq I(t, x, \beta)(\tau)$. Consequently,

$$I(t, x, \beta)(\tau) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} (J(t, x; u, \beta(u))(\tau)), \text{ P-a.s.}$$

(4). Similarly to the above proof, we can prove

$$(\operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} I(t, x, \beta))(\tau) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} (I(t, x, \beta)(\tau)), \text{ P-a.s.}$$

(5). $W(t, x)$ is invariant with respect to the transformation τ, i.e.,

$$W(t, x)(\tau) = W(t, x), \text{ P-a.s.}$$

Indeed, combing those steps above we have

$$\begin{aligned} W(t, x)(\tau) &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} (J(t, x; u, \beta(u))(\tau)) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u(\tau), \hat{\beta}(u(\tau))) \\ &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \hat{\beta}(u)) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) \\ &= W(t, x), \text{ P-a.s.} \end{aligned}$$

where in the both latter equalities we have used $\{u(\tau) | u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T},\ \{\hat{\beta} | \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$.

□

Now let $l \geq 1$. We define the transformation $\tau'_l : \Omega_1 \to \Omega_1$ such that, for all $\omega_1 \in \Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$,

$$\begin{aligned} &(\tau'_l \omega_1)((t - l, r]) = \omega_1((t - 2l, r - l])(:= \omega_1(r - l) - \omega_1(t - 2l)); \\ &(\tau'_l \omega_1)((t - 2l, r - l]) = \omega_1((t - l, r]), \text{ for } r \in [t - l, t]; \\ &(\tau'_l \omega_1)((s, r]) = \omega_1((s, r]),\ (s, r] \cap (t - 2l, t] = \emptyset; \\ &(\tau'_l \omega_1)(0) = 0. \end{aligned} \tag{3.7}$$

Moreover, for $p \in \Omega_2,\ p = \sum_{x \in D_p} p(x) \delta_x$, we set

$$\tau''_l := \sum_{x \in D_p \cap (t-2l, t]^c} p(x) \delta_x + \sum_{x \in D_p \cap (t-l, t]} p(x) \delta_{x-l} + \sum_{x \in D_p \cap (t-2l, t-l]} p(x) \delta_{x+l}.$$

It is easy to check that, $\tau''_l : \Omega_2 \to \Omega_2$ is a bijection, $\tau''^{-1}_l = \tau''_l$, which preserves the measure $P_2 \circ [\tau'']^{-1} = P_2$. Moreover,

$$\begin{aligned} &\mu(\tau''_l p; (t - l, r] \times \Delta) = \mu(p; (t - 2l, r - l] \times \Delta),\ r \in (t - l, t],\ \Delta \in \mathcal{B}(E); \\ &\mu(\tau''_l p; (t - 2l, r - l] \times \Delta) = \mu(p; (t - l, r] \times \Delta),\ r \in (t - l, t],\ \Delta \in \mathcal{B}(E); \\ &\mu(\tau''_l p; (s, r] \times \Delta) = \mu(p; (s, r] \times \Delta),\ (s, r] \cap (t - 2l, t] = \emptyset,\ \Delta \in \mathcal{B}(E). \end{aligned} \tag{3.8}$$

Consequently, the transformation $\tau_l : \Omega \to \Omega,\ \tau_l\omega := (\tau_l'\omega_1, \tau_l''p),\ \omega = (\omega_1, p) \in \Omega = \Omega_1 \times \Omega_2$, satisfies the assumptions (i), (ii), (iii) of Lemma 3.1. Therefore, $W(t,x)(\tau_l) = W(t,x)$, P-a.s., $l \geq 1$. Combined with the following auxiliary Lemma 3.2, we can complete the proof of Proposition 3.1.

Lemma 3.2. *Let $\zeta \in L^\infty(\Omega, \mathcal{F}_t, P)$ be such that, for all $l \geq 1$ natural number, $\zeta(\tau_l) = \zeta$, P-a.e. Then, there exists some real C such that $\zeta = C$, P-a.s.*

For the proof the reader is referred to Lemma 3.2 in Buckdahn, Li and Hu [6].

From (3.2) and (3.5)-(the definition of the value function $W(t,x)$), we get the following property:

Lemma 3.3. *There exists a constant $C > 0$ such that, for all $0 \leq t \leq T,\ x, x' \in \mathbb{R}$,*

$$\begin{array}{ll} \text{(i)} & |W(t,x) - W(t,x')| \leq C|x - x'|; \\ \text{(ii)} & |W(t,x)| \leq C(1 + |x|). \end{array} \tag{3.9}$$

Moreover, for $W(t,x)$, we have the following monotonic property.

Lemma 3.4. *Under the assumptions* (**H2.2**), (**H2.3**)*, the cost functional $J(t,x;u,v)$, for any $u \in \mathcal{U}_{t,T},\ v \in \mathcal{V}_{t,T}$, and the value function $W(t,x)$ are monotonic in the following sense: for each $x, \bar{x} \in \mathbb{R},\ t \in [0,T]$,*

$$\begin{array}{l} \text{(i) } \langle J(t,x;u,v) - J(t,\bar{x};u,v), G(x - \bar{x})\rangle \geq 0,\ P\text{-a.s.}; \\ \text{(ii) } \langle W(t,x) - W(t,\bar{x}), G(x - \bar{x})\rangle \geq 0. \end{array}$$

Proof. We define $(\hat{X}_s, \hat{Y}_s, \hat{Z}_s, \hat{K}_s) := (X_s^{t,x;u,v} - X_s^{t,\bar{x};u,v}, Y_s^{t,x;u,v} - Y_s^{t,\bar{x};u,v}, Z_s^{t,x;u,v} - Z_s^{t,\bar{x};u,v}, K_s^{t,x;u,v} - K_s^{t,\bar{x};u,v})$, and $\Delta l(s) = l(s, \Pi_s^{t,x;u,v}, K_s^{t,x;u,v}, u_s, v_s) - l(s, \Pi_s^{t,\bar{x};u,v}, K_s^{t,\bar{x};u,v}, u_s, v_s)$, for $l = b, \sigma, f, A$, respectively, and $\Delta h(s,e) = h(s, \Pi_{s-}^{t,x;u,v}, u_s, v_s, e) - h(s, \Pi_{s-}^{t,\bar{x};u,v}, u_s, v_s, e)$. Applying Itô's formula to $\langle \hat{Y}_s, G\hat{X}_s\rangle$, we get immediately from (**H2.3**),

$$\begin{array}{rl} & \langle J(t,x;u,v) - J(t,\bar{x};u,v), G(x - \bar{x})\rangle = E[\langle Y_t^{t,x;u,v} - Y_t^{t,\bar{x};u,v}, G(x - \bar{x})\rangle | \mathcal{F}_t] \\ = & E[\langle \Phi(X_T^{t,x;u,v}) - \Phi(X_T^{t,\bar{x};u,v}), G\hat{X}_T\rangle - \int_t^T \langle \Delta A(r), (\hat{X}_r, \hat{Y}_r, \hat{Z}_r)\rangle dr - \int_t^T \int_E \langle G\Delta h(r,e), \hat{K}_r(e)\rangle \lambda(de) dr | \mathcal{F}_t] \\ \geq & E[\mu_1 |G\hat{X}_T|^2 + \int_t^T (\beta_1 |G\hat{X}_r|^2 + \beta_2(|G^T\hat{Y}_r|^2 + |G^T\hat{Z}_r|^2) + \beta_3 \int_E |G^T\hat{K}_r(e)|^2 \lambda(de)) dr | \mathcal{F}_t] \\ \geq & 0, \text{ for any } u \in \mathcal{U}_{t,T},\ v \in \mathcal{V}_{t,T}. \end{array}$$

From Remark 6.1, $W(t,x) = \inf\limits_{\beta \in \mathcal{B}_{t,T}} \sup\limits_{u \in \mathcal{U}_{t,T}} E[J(t,x;u,\beta(u))]$. Setting $V(t,x,\beta) = \sup\limits_{u \in \mathcal{U}_{t,T}} E[J(t,x;u,\beta(u))]$, we always have $V(t,x,\beta) \geq E[J(t,x;u,\beta(u))]$, for any $u \in \mathcal{U}_{t,T}$. On the other hand, for any $\varepsilon > 0$, there exists $u^\varepsilon \in \mathcal{U}_{t,T}$, such that $V(t,\bar{x},\beta) \leq E[J(t,\bar{x};u^\varepsilon,\beta(u^\varepsilon))] + \varepsilon$.
If $G(x - \bar{x}) \geq 0$, then

$$\begin{array}{rcl} \langle V(t,x,\beta) - V(t,\bar{x},\beta), G(x - \bar{x})\rangle & \geq & \langle E[J(t,x;u^\varepsilon,\beta(u^\varepsilon)) - J(t,\bar{x};u^\varepsilon,\beta(u^\varepsilon))] - \varepsilon, G(x - \bar{x})\rangle \\ & \geq & -\varepsilon G(x - \overline{x}). \end{array}$$

If $G(x - \bar{x}) \leq 0$, then similarly, there exists $u^\varepsilon \in \mathcal{U}_{t,T}$ such that $V(t,x,\beta) \leq E[J(t,x;u^\varepsilon,\beta(u^\varepsilon))] + \varepsilon$,

$$\begin{array}{rcl} \langle V(t,x,\beta) - V(t,\bar{x},\beta), G(x - \bar{x})\rangle & = & \langle V(t,\bar{x},\beta) - V(t,x,\beta), G(\bar{x} - x)\rangle \\ & \geq & \langle E[J(t,\bar{x};u^\varepsilon,\beta(u^\varepsilon)) - J(t,x;u^\varepsilon,\beta(u^\varepsilon))] - \varepsilon, G(\bar{x} - x)\rangle \\ & \geq & -\varepsilon G(\bar{x} - x). \end{array}$$

Therefore, we always have $\langle V(t,x,\beta) - V(t,\bar{x},\beta), G(x - \bar{x})\rangle \geq -\varepsilon|G(x - \overline{x})|$, for any $\beta \in \mathcal{B}_{t,T},\ x, \bar{x} \in \mathbb{R},\ t \in [0,T]$. Since $W(t,x) = \inf\limits_{\beta \in \mathcal{B}_{t,T}} \sup\limits_{u \in \mathcal{U}_{t,T}} E[J(t,x;u,\beta(u))] = \inf\limits_{\beta \in \mathcal{B}_{t,T}} V(t,x,\beta)$, we have $W(t,x) \leq V(t,x,\beta)$, for any $\beta \in \mathcal{B}_{t,T}$. Moreover, for any $\varepsilon > 0$, there exists $\beta^\varepsilon \in \mathcal{B}_{t,T}$ such that $W(t,x) + \varepsilon \geq V(t,x,\beta^\varepsilon)$.
If $G(x - \bar{x}) \geq 0$, then

$$\langle W(t,x) - W(t,\bar{x}), G(x - \bar{x})\rangle \ \ \geq \ \ \langle V(t,x,\beta^\varepsilon) - V(t,\bar{x},\beta^\varepsilon) - \varepsilon, G(x - \bar{x})\rangle \geq -2\varepsilon G(x - \overline{x}).$$

If $G(x - \bar{x}) \leq 0$, then with $\beta^\varepsilon \in \mathcal{B}_{t,T}$ such that $W(t,\bar{x}) + \varepsilon \geq V(t,\bar{x},\beta^\varepsilon)$,

$$\begin{array}{rcl} \langle W(t,x) - W(t,\bar{x}), G(x - \bar{x})\rangle & = & \langle W(t,\bar{x}) - W(t,x), G(\bar{x} - x)\rangle \\ & \geq & \langle V(t,\bar{x},\beta^\varepsilon) - V(t,x,\beta^\varepsilon) - \varepsilon, G(\bar{x} - x)\rangle \geq -2\varepsilon G(\bar{x} - x). \end{array}$$

Therefore, we have $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle \geq -2\varepsilon|G(x-\overline{x})|$, for any $x,\bar{x}\in\mathbb{R},\ t\in[0,T]$. Letting $\varepsilon\downarrow 0$, $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle \geq 0$, for any $x,\ \bar{x}\in\mathbb{R},\ t\in[0,T]$.

$\square$

Remark 3.5. *(1) From* (**H2.3**)-(i) *we know that, if σ doesn't depend on z, then $\beta_2=0$; if h doesn't depend on k, then $\beta_3=0$. Furthermore, we assume:*

(**H3.1**) (i) *The Lipschitz constant $L_\sigma\geq 0$ of σ with respect to z is sufficiently small, i.e., there exists some $L_\sigma\geq 0$ small enough such that, for all $t\in[0,T],\ u\in U,\ v\in V,\ x_1,x_2\in\mathbb{R},\ y_1,y_2\in\mathbb{R},\ z_1,z_2\in\mathbb{R}^d$,*

$$|\sigma(t,x_1,y_1,z_1,u,v)-\sigma(t,x_2,y_2,z_2,u,v)|\leq K(|x_1-x_2|+|y_1-y_2|)+L_\sigma|z_1-z_2|.$$

And the Lipschitz coefficient $L_h(\cdot)$ of h with respect to z is sufficiently small, i.e., there exists a function $L_h:E\to\mathbb{R}^+$ with $\tilde{C}_h:=\max(\sup_{e\in E}L_h^2(e),\int_E L_h^2(e)\lambda(de))<+\infty$ sufficiently small, and for all $t\in[0,T],\ u\in U,\ v\in V,\ x_1,x_2\in\mathbb{R},\ y_1,y_2\in\mathbb{R},\ z_1,z_2\in\mathbb{R}^d,\ e\in E$,

$$|h(t,x_1,y_1,z_1,u,v,e)-h(t,x_2,y_2,z_2,u,v,e)|\leq \rho(e)(|x_1-x_2|+|y_1-y_2|)+L_h(e)|z_1-z_2|.$$

(ii) *For all $t\in[0,T],\ u\in U,\ v\in V$, for any $(x,y,z)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$, P-a.s., $|h(t,x,y,z,u,v,e)|\leq \rho(e)(1+|x|+|y|)$, where $\rho(e)=C(1\wedge|e|)$.*

(2) Notice that when $\sigma,\ h$ don't depend on z, it's clearly that (**H3.1**) *always holds true.*

Now we adopt Peng's notion of stochastic backward semigroup to discuss a generalized DPP for our stochastic differential game (3.1), (3.5). The notation of stochastic backward semigroup was first introduced by Peng [30] to prove the DPP for stochastic control problems. Similar to [18], first we define the family of (backward) semigroups associated with FBSDE with jumps (3.1).

For given initial data (t,x), a number $0<\delta\leq T-t$, admissible control processes $u(\cdot)\in\mathcal{U}_{t,t+\delta},\ v(\cdot)\in\mathcal{V}_{t,t+\delta}$ and a real-valued random function $\Psi:\Omega\times\mathbb{R}\to\mathbb{R}$, $\mathcal{F}_{t+\delta}\otimes\mathcal{B}(\mathbb{R})$-measurable such that (**H2.3**)-(ii) holds, we put

$$G^{t,x;u,v}_{s,t+\delta}[\Psi(t+\delta,\overline{X}^{t,x;u,v}_{t+\delta})]:=\overline{Y}^{t,x;u,v}_s,\ \ s\in[t,t+\delta],$$

where $(\overline{\Pi}^{t,x;u,v}_s,\overline{K}^{t,x;u,v}_s):=(\overline{X}^{t,x;u,v}_s,\overline{Y}^{t,x;u,v}_s,\overline{Z}^{t,x;u,v}_s,\overline{K}^{t,x;u,v}_s)_{t\leq s\leq t+\delta}$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{rcl} d\overline{X}^{t,x;u,v}_s &=& b(s,\overline{\Pi}^{t,x;u,v}_s,u_s,v_s)ds+\sigma(s,\overline{\Pi}^{t,x;u,v}_s,u_s,v_s)dB_s+\int_E h(s,\overline{\Pi}^{t,x;u,v}_{s-},u_s,v_s,e)\tilde{\mu}(ds,de),\\ d\overline{Y}^{t,x;u,v}_s &=& -f(s,\overline{\Pi}^{t,x;u,v}_s,\int_E\overline{K}^{t,x;u,v}_s(e)l(e)\lambda(de),u_s,v_s)ds+\overline{Z}^{t,x;u,v}_s dB_s+\int_E\overline{K}^{t,x;u,v}_s(e)\tilde{\mu}(ds,de),\\ \overline{X}^{t,x;u,v}_t &=& x, \qquad\qquad s\in[t,t+\delta],\\ \overline{Y}^{t,x;u,v}_{t+\delta} &=& \Psi(t+\delta,\overline{X}^{t,x;u,v}_{t+\delta}). \end{array}\right. \tag{3.10}$$

Here we write again $\overline{\Pi}^{t,x;u,v}_{s-}=(\overline{X}^{t,x;u,v}_{s-},\overline{Y}^{t,x;u,v}_{s-},\overline{Z}^{t,x;u,v}_s)$.

Remark 3.6. *From Theorem 3.2 in [19], we know FBSDE (3.10) has a unique solution $(\overline{X}^{t,x;u,v},\overline{Y}^{t,x;u,v},\overline{Z}^{t,x;u,v},\overline{K}^{t,x;u,v})$ on the small interval $[t,t+\delta]$, for any $0\leq\delta\leq\delta_0$, where $\delta_0>0$ is independent of (t,x) and the controls $u,\ v$.*

Then, for the solution $(X^{t,x;u,v},Y^{t,x;u,v},Z^{t,x;u,v},K^{t,x;u,v})$ of FBSDE (3.1) we get

$$G^{t,x;u,v}_{t,T}[\Phi(X^{t,x;u,v}_T)]=G^{t,x;u,v}_{t,t+\delta}[Y^{t,x;u,v}_{t+\delta}].$$

We also have

$$J(t,x;u,v)=Y^{t,x;u,v}_t=G^{t,x;u,v}_{t,T}[\Phi(X^{t,x;u,v}_T)]=G^{t,x;u,v}_{t,t+\delta}[Y^{t,x;u,v}_{t+\delta}]=G^{t,x;u,v}_{t,t+\delta}[J(t+\delta,X^{t,x;u,v}_{t+\delta};u,v)].$$

Theorem 3.1. *Under the assumptions* (**H2.2**), (**H2.3**) *and* (**H3.1**)*, the lower value function $W(t,x)$ satisfies the following DPP: there exists sufficiently small $\delta_0>0$ such that, for any $0\le\delta\le\delta_0,\ t\in[0,T-\delta],\ x\in\mathbb{R}$,*

$$W(t,x)=\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u,\beta(u)}_{t+\delta})].$$

The proof is given in the Appendix.

From the definition of our stochastic backward semigroup we know here:

$$G^{t,x;u,v}_{s,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u,v}_{t+\delta})]:=\widetilde{Y}^{t,x;u,v}_s,\ s\in[t,t+\delta],\ u(\cdot)\in\mathcal{U}_{t,t+\delta},\ v(\cdot)\in\mathcal{V}_{t,t+\delta},$$

where $(\widetilde{\Pi}^{t,x;u,v}_s,\widetilde{K}^{t,x;u,v}_s)_{t\le s\le t+\delta}:=(\widetilde{X}^{t,x;u,v}_s,\widetilde{Y}^{t,x;u,v}_s,\widetilde{Z}^{t,x;u,v}_s,\widetilde{K}^{t,x;u,v}_s)_{t\le s\le t+\delta}$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{rcl} d\widetilde{X}^{t,x;u,v}_s &=& b(s,\widetilde{\Pi}^{t,x;u,v}_s,u_s,v_s)ds+\sigma(s,\widetilde{\Pi}^{t,x;u,v}_s,u_s,v_s)dB_s+\int_E h(s,\widetilde{\Pi}^{t,x;u,v}_{s-},u_s,v_s,e)\tilde{\mu}(ds,de),\\ d\widetilde{Y}^{t,x;u,v}_s &=& -f(s,\widetilde{\Pi}^{t,x;u,v}_s,\int_E\widetilde{K}^{t,x;u,v}_s(e)l(e)\lambda(de),u_s,v_s)ds+\widetilde{Z}^{t,x;u,v}_s dB_s+\int_E\widetilde{K}^{t,x;u,v}_s(e)\tilde{\mu}(ds,de),\\ \widetilde{X}^{t,x;u,v}_t &=& x, \qquad\qquad s\in[t,t+\delta],\\ \widetilde{Y}^{t,x;u,v}_{t+\delta} &=& W(t+\delta,\widetilde{X}^{t,x;u,v}_{t+\delta}). \end{array}\right. \tag{3.11}$$

From Lemma 3.3 we get that the value function $W(t,x)$ is Lipschitz continuous in x, uniformly in t. Now with the help of DPP we can derive the $\frac{1}{2}$-Hölder continuity property of $W(t,x)$ in t.

Theorem 3.2. *Under the assumptions* (**H2.2**), (**H2.3**), (**H3.1**)*, the lower value function $W(t,x)$ is $\frac{1}{2}$-Hölder continuous in t: there exists a constant C such that, for all $x\in\mathbb{R},\ t,t'\in[0,T]$,*

$$|W(t,x)-W(t',x)|\le C(1+|x|)|t-t'|^{\frac{1}{2}}.$$

Proof. Let $(t,x)\in[0,T]\times\mathbb{R}$, and $0<\delta\le(T-t)\wedge\delta_0$. Obviously, for the desired result, it is sufficient to prove the following inequality: for some constant C,

$$-C(1+|x|)\delta^{\frac{1}{2}}\le W(t,x)-W(t+\delta,x)\le C(1+|x|)\delta^{\frac{1}{2}}.$$

Next we only prove the second inequality. From Remark 6.1, we know that for every $\beta\in\mathcal{B}_{t,t+\delta}$ there exists $u^\varepsilon\in\mathcal{U}_{t,t+\delta}$, such that

$$W(t,x)\le G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta})]+\varepsilon.$$

Therefore, $W(t,x)-W(t+\delta,x)\le G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta})]+\varepsilon-W(t+\delta,x)=I^1_\delta+I^2_\delta+\varepsilon$, where

$$\begin{array}{rcl} I^1_\delta &=& G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta})]-G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta,x)],\\ I^2_\delta &=& G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta,x)]-W(t+\delta,x). \end{array}$$

We know $G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{s,t+\delta}[W(t+\delta,x)]:=\hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s,\ s\in[t,t+\delta],\ \beta\in\mathcal{B}_{t,t+\delta}$, where $(\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\hat{K}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}):=(\hat{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\hat{Z}^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\hat{K}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{rcl} d\hat{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s &=& b(s,\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s,u^\varepsilon_s,\beta_s(u^\varepsilon))ds+\sigma(s,\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s,u^\varepsilon_s,\beta_s(u^\varepsilon))dB_s\\ && +\int_E h(s,\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{s-},u^\varepsilon_s,\beta_s(u^\varepsilon),e)\tilde{\mu}(ds,de),\\ d\hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s &=& -f(s,\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s,\int_E\hat{K}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s(e)l(e)\lambda(de),u^\varepsilon_s,\beta_s(u^\varepsilon))ds\\ && +\hat{Z}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s dB_s+\int_E\hat{K}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s(e)\tilde{\mu}(ds,de),\qquad s\in[t,t+\delta],\\ \hat{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_t &=& x,\\ \hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta} &=& W(t+\delta,x), \end{array}\right. \tag{3.12}$$

where $\hat{\Pi}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{s-}=(\hat{X}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{s-},\hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{s-},\hat{Z}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s)$.

By applying Itô's formula to $e^{\gamma s}|\widetilde{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s-\hat{Y}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_s|^2$, taking γ large enough and using standard

methods for BSDEs, we deduce from Lemma 3.3, Theorem 3.4-(ii) and Remark 3.7 in [19]

$$\begin{array}{rl} & |\widetilde{Y}_t^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - \hat{Y}_t^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|^2 \\ \leq & CE[|W(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - W(t+\delta,x)|^2|\mathcal{F}_t] + CE[\int_t^{t+\delta}|\widetilde{X}_r^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - \hat{X}_r^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|^2dr|\mathcal{F}_t] \\ \leq & CE[|\widetilde{X}_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] \\ & +C\delta(E[\sup\limits_{r\in[t,t+\delta]}|\widetilde{X}_r^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] + E[\sup\limits_{r\in[t,t+\delta]}|\hat{X}_r^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t]) \\ \leq & C\delta(1+|x|^2). \end{array} \tag{3.13}$$

Therefore, there exists some constant C independent of the controls such that

$$|I_\delta^1| = |\widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - \hat{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| \leq C\delta^{\frac{1}{2}}(1+|x|).$$

Due to the definition of $G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[\cdot]$, we can rewrite the second term I_δ^2 as follows

$$\begin{array}{rcl} |I_\delta^2| & = & |E[\int_t^{t+\delta} f(s,\hat{\Pi}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\int_E \hat{K}_s^{t,x;u,v}(e)l(e)\lambda(de),u_s^\varepsilon,\beta_s(u^\varepsilon))ds \mid \mathcal{F}_t]| \\ & \leq & \delta^{\frac{1}{2}}E[\int_t^{t+\delta}|f(s,\hat{\Pi}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)},\int_E \hat{K}_s^{t,x;u,v,\beta(u^\varepsilon)}(e)l(e)\lambda(de),u_s^\varepsilon,\beta_s(u^\varepsilon))|^2ds \mid \mathcal{F}_t]^{\frac{1}{2}} \\ & \leq & C\delta^{\frac{1}{2}}E[\int_t^{t+\delta}(1+|\hat{X}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| + |\hat{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| + |\hat{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| \\ & & +\int_E|\hat{K}_s^{t,x;u,\beta(u^\varepsilon)}(e)|l(e)\lambda(de))^2ds \mid \mathcal{F}_t]^{\frac{1}{2}} \\ & \leq & C(1+|x|)\delta^{\frac{1}{2}}. \end{array}$$

For the latter inequality, we have used estimates (refer to Remark 3.7 in [19]) for FBSDEs (3.12) with jumps. Therefore, $W(t,x) - W(t+\delta,x) \leq C(1+|x|)\delta^{\frac{1}{2}} + \varepsilon$. Letting $\varepsilon \downarrow 0$, we complete the proof.

□

4 Viscosity Solutions of Isaacs' equations with integral-differential operators

Now we consider the following fully coupled FBSDE with jumps:

$$\left\{\begin{array}{rcl} dX_s^{t,x;u,v} & = & b(s,\Pi_s^{t,x;u,v},u_s,v_s)ds + \sigma(s,\Pi_s^{t,x;u,v},u_s,v_s)dB_s + \int_E h(s,\Pi_{s-}^{t,x;u,v},u_s,v_s,e)\tilde{\mu}(ds,de), \\ dY_s^{t,x;u,v} & = & -f(s,\Pi_s^{t,x;u,v},\int_E K_s^{t,x;u,v}(e)l(e)\lambda(de),u_s,v_s)ds + Z_s^{t,x;u,v}dB_s + \int_E K_s^{t,x;u,v}(e)\tilde{\mu}(ds,de), \\ X_t^{t,x;u,v} & = & x, \qquad s\in[t,T], \\ Y_T^{t,x;u,v} & = & \Phi(X_T^{t,x;u,v}), \end{array}\right. \tag{4.1}$$

where $\Pi_{s-}^{t,x;u,v} = (X_{s-}^{t,x;u,v}, Y_{s-}^{t,x;u,v}, Z_s^{t,x;u,v})$, and the related second order integral-partial differential equations of Isaacs' type which are the following PDEs combined with an algebraic equation:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}W(t,x) + H_V^-(t,x,W(t,x)) = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}, \\ V(t,x,u,v) = DW(t,x).\sigma(t,x,W(t,x),V(t,x,u,v),u,v), \\ W(T,x) = \Phi(x), \qquad x\in\mathbb{R}, \end{array}\right. \tag{4.2}$$

and

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}U(t,x) + H_{V_1}^+(t,x,U(t,x)) = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}, \\ V_1(t,x,u,v) = DU(t,x).\sigma(t,x,U(t,x),V_1(t,x,u,v),u,v), \\ U(T,x) = \Phi(x), \qquad x\in\mathbb{R}, \end{array}\right. \tag{4.3}$$

where

$$\begin{array}{rcl} H_V^-(t,x,W(t,x)) & = & \sup\limits_{u\in U}\inf\limits_{v\in V} H(t,x,W(t,x),DW(t,x),D^2W(t,x),V(t,x,u,v),u,v), \\ H_{V_1}^+(t,x,U(t,x)) & = & \inf\limits_{v\in V}\sup\limits_{u\in U} H(t,x,U(t,x),DU(t,x),D^2U(t,x),V_1(t,x,u,v),u,v), \end{array}$$

and

$$\begin{array}{l} H(t,x,\phi,p,A,r,u,v) \\ = \frac{1}{2}tr(\sigma\sigma^T(t,x,\phi(t,x),r,u,v)A) + p.b(t,x,\phi(t,x),r,u,v) \\ \quad +\int_E[\phi(t,x+h(t,x,\phi(t,x),r,u,v,e)) - \phi(t,x) - p.h(t,x,\phi(t,x),r,u,v,e)]\lambda(de) \\ \quad +f(t,x,\phi(t,x),r,\int_E[\phi(t,x+h(t,x,\phi(t,x),r,u,v,e)) - \phi(t,x)]l(e)\lambda(de),u,v), \end{array}$$

where $t \in [0,T],\ x \in \mathbb{R},\ \phi \in C([0,T]\times\mathbb{R},\mathbb{R}),\ p \in \mathbb{R}^d,\ A \in \mathcal{S}^d,\ r \in \mathbb{R}^d,\ u \in U,\ v \in V$, where $\mathcal{S}^d$ is the set of $d \times d$ symmetric matrices.

We will show that the value function $W(t,x)$ (resp., $U(t,x)$) defined in (3.5) (resp., (3.6)) is a viscosity solution of the corresponding equation (4.2) (resp., (4.3)). For this we use Peng's BSDE approach [30] developed originally for stochastic control problems of decoupled FBSDEs. We first give the definition of viscosity solution for this kind of PDEs. For more information on viscosity solution, the reader is referred to Crandall, Ishii and Lions [7].

Definition 4.1. *A real-valued continuous function $W \in C([0,T]\times\mathbb{R})$ is called*
(i) *a viscosity subsolution of equation (4.2) if $W(T,x) \le \Phi(x)$, for all $x \in \mathbb{R}$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R})$ and for all $(t,x) \in [0,T)\times\mathbb{R}$ such that $W-\varphi$ attains a local maximum at (t,x),*

$$\begin{array}{l}\frac{\partial\varphi}{\partial t}(t,x) + \sup\limits_{u\in U}\inf\limits_{v\in V}\{A^{u,v}\varphi(t,x) + B^{\delta,u,v}(W,\varphi)(t,x)\\ +f(t,x,W(t,x),\psi(t,x,u,v),C^{\delta,u,v}(W,\varphi)(t,x),u,v)\} \ge 0, \text{ for any } \delta>0, \text{ and}\\ \psi(t,x,u,v) = D\varphi(t,x).\sigma(t,x,W(t,x),\psi(t,x,u,v),u,v),\end{array} \tag{4.4}$$

where

$$\begin{array}{rl}A^{u,v}\varphi(t,x) = & \frac{1}{2}tr(\sigma\sigma^T(t,x,W(t,x),\psi(t,x,u,v),u,v)D^2\varphi(t,x)) + D\varphi(t,x).b(t,x,W(t,x),\psi(t,x,u,v),u,v),\\ B^{\delta,u,v}(W,\varphi)(t,x) = & \int_{E_\delta}(\varphi(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - \varphi(t,x)\\ & \qquad -D\varphi(t,x).h(t,x,W(t,x),\psi(t,x,u,v),u,v,e))\lambda(de)\\ & +\int_{E^c_\delta}(W(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - W(t,x)\\ & \qquad -D\varphi(t,x).h(t,x,W(t,x),\psi(t,x,u,v),u,v,e))\lambda(de)\\ C^{\delta,u,v}(W,\varphi)(t,x) = & \int_{E_\delta}(\varphi(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - \varphi(t,x))l(e)\lambda(de)\\ & +\int_{E^c_\delta}(W(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - W(t,x))l(e)\lambda(de)\end{array}$$

with $E_\delta = \{e \in E \,||e| < \delta\}$.
(ii) *a viscosity supersolution of equation (4.2) if $W(T,x) \ge \Phi(x)$, for all $x \in \mathbb{R}$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R})$ and for all $(t,x) \in [0,T)\times\mathbb{R}$ such that $W-\varphi$ attains a local minimum at (t,x),*

$$\begin{array}{l}\frac{\partial\varphi}{\partial t}(t,x) + \sup\limits_{u\in U}\inf\limits_{v\in V}\{A^{u,v}\varphi(t,x) + B^{\delta,u,v}(W,\varphi)(t,x)\\ +f(t,x,W(t,x),\psi(t,x,u,v),C^{\delta,u,v}(W,\varphi)(t,x),u,v)\} \le 0, \text{ for any } \delta>0, \text{ and}\\ \psi(t,x,u,v) = D\varphi(t,x).\sigma(t,x,W(t,x),\psi(t,x,u,v),u,v),\end{array}$$

(iii) *a viscosity solution of equation (4.2) if it is both a viscosity sub- and supersolution of equation (4.2).*

Similar to the results in [2, 6, 17], we claim the following result.

Lemma 4.1. *In Definition 4.1, we can replace $B^{\delta,u,v}(W,\varphi)(t,x)$ and $C^{\delta,u,v}(W,\varphi)(t,x)$ by $B^{u,v}\varphi(t,x)$ and $C^{u,v}\varphi(t,x)$, respectively, where*

$$\begin{array}{rl}B^{u,v}\varphi(t,x) = & \int_E(\varphi(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - \varphi(t,x)\\ & -D\varphi(t,x).h(t,x,W(t,x),\psi(t,x,u,v),u,v,e))\lambda(de),\\ C^{u,v}\varphi(t,x) = & \int_E(\varphi(t,x+h(t,x,W(t,x),\psi(t,x,u,v),u,v,e)) - \varphi(t,x))l(e)\lambda(de).\end{array}$$

In what follows, we always assume $W(t,x) = \varphi(t,x)$, otherwise, we can replace φ by $\varphi-(W(t,x)-\varphi(t,x))$. From now on, we shall use the following equivalent definition of viscosity solution.

Definition 4.2. *A real-valued continuous function $W \in C([0,T]\times\mathbb{R})$ is called*
(i) *a viscosity subsolution of equation (4.2) if $W(T,x) \le \Phi(x)$, for all $x \in \mathbb{R}$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R})$ and for all $(t,x) \in [0,T)\times\mathbb{R}$ such that $W-\varphi$ attains a local maximum at (t,x),*

$$\begin{cases}\frac{\partial\varphi}{\partial t}(t,x) + H^-_\psi(t,x,\varphi(t,x)) \ge 0,\\ \text{where } \psi \text{ is the unique solution of the following algebraic equation:}\\ \psi(t,x,u,v) = D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x,u,v),u,v).\end{cases}$$

(ii) *a viscosity supersolution of equation (4.2) if* $W(T,x) \geq \Phi(x)$, *for all* $x \in \mathbb{R}$, *and if for all functions* $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R})$ *and for all* $(t,x) \in [0,T)\times\mathbb{R}$ *such that* $W-\varphi$ *attains a local minimum at* (t,x),

$$\begin{cases} \frac{\partial\varphi}{\partial t}(t,x) + H^-_\psi(t,x,\varphi(t,x)) \leq 0, \\ \text{where } \psi \text{ is the unique solution of the following algebraic equation:} \\ \psi(t,x,u,v) = D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x,u,v),u,v). \end{cases}$$

(iii) *a viscosity solution of equation (4.2) if it is both a viscosity sub- and supersolution of equation (4.2).*

Remark 4.1. *When* σ *depends on* z, *we need the test function* φ *in Definitions 4.1 and 4.2 satisfies the monotonicity condition* (**H2.3**)-(ii)*' and the following technical assumptions:*

(**H4.1**) (i) $\beta_2 > 0$;

(ii) $G\sigma(s,x,y,z,u,v)$ *is continuous in* (s,u,v), *uniformly with respect to* $(x,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$.

Theorem 4.1. *Under the assumptions* (**H2.2**), (**H2.3**), (**H3.1**), (**H4.1**), *the lower value function* W *is a viscosity solution of (4.2), the upper value function* U *is a viscosity solution of (4.3).*

We only give the proof for W, similar to U. Before proving the theorem, we first consider the following equation:

$$\begin{cases} dX^{u,v}_s &= b(s,\Pi^{u,v}_s,u_s,v_s)ds + \sigma(s,\Pi^{u,v}_s,u_s,v_s)dB_s + \int_E h(s,\Pi^{u,v}_{s-},u_s,v_s,e)\tilde{\mu}(ds,de), \quad s\in[t,t+\delta], \\ dY^{u,v}_s &= -f(s,\Pi^{u,v}_s,\int_E K^{u,v}_s(e)l(e)\lambda(de),u_s,v_s)ds + Z^{u,v}_s dB_s + \int_E K^{u,v}_s(e)\tilde{\mu}(ds,de), \\ X^{u,v}_t &= x, \\ Y^{u,v}_{t+\delta} &= \varphi(t+\delta, X^{u,v}_{t+\delta}), \quad 0\leq\delta\leq T-t, \end{cases} \tag{4.5}$$

where $\Pi^{u,v}_{s-} = (X^{u,v}_{s-},Y^{u,v}_{s-},Z^{u,v}_s)$. From Theorems 3.2 and 3.4 in [19], we know there exists $0<\bar{\delta}_0\leq T-t$ such that for any $0\leq\delta\leq\bar{\delta}_0$, (4.5) has a unique solution $(\Pi^{u,v}_s,K^{u,v}_s)_{s\in[t,t+\delta]} := (X^{u,v}_s,Y^{u,v}_s,Z^{u,v}_s,K^{u,v}_s)_{s\in[t,t+\delta]} \in \mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)\times\mathcal{K}^2_\lambda(t,t+\delta;\mathbb{R})$, and for $p\geq 2$,

$$\begin{array}{ll} \text{(i)} & E[\sup\limits_{t\leq s\leq t+\delta}|X^{u,v}_s|^p + \sup\limits_{t\leq s\leq t+\delta}|Y^{u,v}_s|^p + (\int_t^{t+\delta}|Z^{u,v}_s|^2ds)^{\frac{p}{2}} \\ & +(\int_t^{t+\delta}\int_E|K^{u,v}_s(e)|^2\lambda(de)ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C(1+|x|^p), \ \text{P-a.s.}; \\ \text{(ii)} & E[\sup\limits_{t\leq s\leq t+\delta}|X^{u,v}_s - x|^p \mid \mathcal{F}_t] \leq C\delta(1+|x|^p), \quad \text{P-a.s.}, \\ \text{(iii)} & E[(\int_t^{t+\delta}|Z^{u,v}_s|^2ds)^{\frac{p}{2}} + (\int_t^{t+\delta}\int_E|K^{u,v}_s(e)|^2\lambda(de)ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C\delta^{\frac{p}{2}}(1+|x|^p), \quad \text{P-a.s.} \end{array} \tag{4.6}$$

Define

$$\begin{array}{rl} & L(s,x,y,z,k,u,v) \\ = & \frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}tr(\sigma\sigma^T(s,x,y+\varphi(s,x),z,u,v)D^2\varphi(s,x)) + D\varphi(s,x).b(s,x,y+\varphi(s,x),z,u,v) \\ & +\int_E[\varphi(s,x+h(s,x,y+\varphi(s,x),z,u,v,e)) - \varphi(s,x) - D\varphi(s,x).h(s,x,y+\varphi(s,x),z,u,v,e)]\lambda(de) \\ & +f(s,x,y+\varphi(s,x),z,\int_E[k(e)+\varphi(s,x+h(s,x,y+\varphi(s,x),z,u,v,e)) - \varphi(s,x)]l(e)\lambda(de),u,v). \end{array}$$

Note that, for all $(s,x,y,z,k,u,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times U\times V$,

$$L(s,x,y,z,k,u,v) \leq C(1+|x|^2+|y|^2+|z|^2+|k|),$$

and for all (s,x,y,z,k,u,v), $(s,\bar{x},\bar{y},\bar{z},\bar{k},u,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times U\times V$,

$$|L(s,x,y,z,k,u,v) - L(s,x,\bar{y},\bar{z},\bar{k},u,v)| \leq C(1+|x|+|y|+|\bar{y}|+|z|+|\bar{z}|)(|y-\bar{y}|+|z-\bar{z}|+|k-\bar{k}|).$$

Now we set $Y^{1,u,v}_s = Y^{u,v}_s - \varphi(s,X^{u,v}_s)$. By applying Itô's formula to $\varphi(s,X^{u,v}_s)$ and setting

$$\begin{array}{l} Z^{1,u,v}_s = Z^{u,v}_s - D\varphi(s,X^{u,v}_s).\sigma(s,X^{u,v}_s,Y^{u,v}_s,Z^{u,v}_s,u_s,v_s), \\ K^{1,u,v}_s(e) = K^{u,v}_s(e) - \varphi(s,X^{u,v}_s + h(s,X^{u,v}_{s-},Y^{u,v}_{s-},Z^{u,v}_s,u_s,v_s,e)) + \varphi(s,X^{u,v}_s), \end{array}$$

we obtain

$$\left\{\begin{array}{rcl} Y_s^{1,u,v} &=& \int_s^{t+\delta}\Big[\frac{\partial}{\partial r}\varphi(r,X_r^{u,v})+\frac{1}{2}tr(\sigma\sigma^T(r,\Pi_r^{u,v},u_r,v_r)D^2\varphi(r,X_r^{u,v}))+D\varphi(r,X_r^{u,v}).b(r,\Pi_r^{u,v},u_r,v_r)\\ && +f(r,\Pi_r^{u,v},\int_E K_r^{u,v}(e)l(e)\lambda(de),u_r,v_r)+\int_E[\varphi(r,X_r^{u,v}+h(r,\Pi_{r-}^{u,v},u_r,v_r,e))-\varphi(r,X_r^{u,v})\\ && -D\varphi(r,X_r^{u,v}).h(r,\Pi_{r-}^{u,v},u_r,v_r,e)]l(e)\lambda(de)\Big]dr-\int_s^{t+\delta}Z_r^{1,u,v}dB_r-\int_s^{t+\delta}\int_E K_r^{1,u,v}(e)\tilde{\mu}(dr,de)\\ &=& \int_s^{t+\delta}L(r,X_r^{u,v},Y_r^{1,u,v},Z_r^{u,v},K_r^{1,u,v},u_r,v_r)dr-\int_s^{t+\delta}Z_r^{1,u,v}dB_r-\int_s^{t+\delta}\int_E K_r^{1,u,v}(e)\tilde{\mu}(dr,de),\\ Z_s^{1,u,v} &=& Z_s^{u,v}-D\varphi(s,X_s^{u,v}).\sigma(s,X_s^{u,v},Y_s^{1,u,v}+\varphi(s,X_s^{u,v}),Z_s^{u,v},u_s,v_s),\quad s\in[t,t+\delta]. \end{array}\right. \tag{4.7}$$

Obviously, (4.7) has a solution $(Y_s^{1,u,v},Z_s^{1,u,v},K_s^{1,u,v})\in\mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)\times\mathcal{K}_\lambda^2(t,t+\delta;\mathbb{R})$, because (4.5) has a unique solution $(X_s^{u,v},Y_s^{u,v},Z_s^{u,v},K_s^{u,v})_{s\in[t,t+\delta]}$, and $Y_s^{1,u,v}=Y_s^{u,v}-\varphi(s,X_s^{u,v})$, $Z_s^{1,u,v}=Z_s^{u,v}-D\varphi(s,X_s^{u,v}).\sigma(s,X_s^{u,v},Y_s^{1,u,v}+\varphi(s,X_s^{u,v}),Z_s^{u,v},u_s,v_s)$, $K_s^{1,u,v}(e)=K_s^{u,v}(e)-\varphi(s,X_s^{u,v}+h(s,X_{s-}^{u,v},Y_{s-}^{u,v},Z_s^{u,v},u_s,v_s,e))+\varphi(s,X_s^{u,v})$.

Next we consider the following BSDE combined with an algebraic equation:

$$\left\{\begin{array}{rcl} dY_s^{2,u,v} &=& -L(s,x,0,\hat{Z}_s^{u,v},0,u_s,v_s)ds+Z_s^{2,u,v}dB_s+\int_E K_s^{2,u,v}(e)\tilde{\mu}(ds,de),\quad s\in[t,t+\delta],\\ \hat{Z}_s^{u,v} &=& Z_s^{1,u,v}+D\varphi(s,x).\sigma(s,x,Y_s^{1,u,v}+\varphi(s,x),\hat{Z}_s^{u,v},u_s,v_s),\\ Y_{t+\delta}^{2,u,v} &=& 0, \end{array}\right. \tag{4.8}$$

where $u(\cdot)\in\mathcal{U}_{t,t+\delta},\ v(\cdot)\in\mathcal{V}_{t,t+\delta}$.

We first recall the following condition (Remark 4.2) and the Representation Theorem for the algebraic equation obtained in [16, 18].

Remark 4.2. *Without loss of generality, we may assume $G=1\in\mathbb{R}$. When σ depends on z, we get the following results directly from the monotonicity condition* (**H2.3**)*:*

$$\begin{array}{ll} \text{(i)} & \langle\sigma(t,x,y,z,u,v)-\sigma(t,x,y,\bar{z},u,v),z-\bar{z}\rangle\le-\beta_2|z-\bar{z}|^2;\\ \text{(ii)} & D\varphi(s,x)\ge0. \end{array} \tag{4.9}$$

Indeed, (i) *follows from* (**H2.3**)-(i), (ii) *follows from* (**H2.3**)-(ii)*' satisfied by* $\varphi:\langle\varphi(s,x)-\varphi(s,\bar{x}),G(x-\bar{x})\rangle\ge0,\ \varphi\in C_{l,b}^3([0,T]\times\mathbb{R})$.

We recall the following Representation Theorem given in [16, 18].

Theorem 4.2. *Under the assumptions* (**H2.3**), (**H3.1**), (**H4.1**), *for any $s\in[0,T],\ \xi\in\mathbb{R}^d,\ x\in\mathbb{R},\ y\in\mathbb{R},\ u\in U,\ v\in V$, there exists a unique z such that $z=\xi+D\varphi(s,x).\sigma(s,x,y+\varphi(s,x),z,u,v)$. That means, the solution z can be written as $z=q(s,x,y,\xi,u,v)$, where the function q is Lipschitz with respect to $y,\ \xi$, and $|q(s,x,y,\xi,u,v)|\le C(1+|x|+|y|+|\xi|)$. The constant C is independent of $s,\ x,\ y,\ \xi,\ u,\ v$. Moreover, $z=q(s,x,y,\xi,u,v)$ is continuous with respect to (s,u,v).*

Remark 4.3. *From the above Representation Theorem we have the existence and the uniqueness of the solutions of the algebraic equations in (4.7) and (4.8).*

In order to complete the proof of Theorem 4.1, we need the following lemmas.

Lemma 4.2. *For every $u\in\mathcal{U}_{t,t+\delta},\ v\in\mathcal{V}_{t,t+\delta},\ 0\le\delta\le\bar{\delta}_0$,*

$$E[\int_t^{t+\delta}(|Y_s^{1,u,v}|+|Z_s^{1,u,v}|+\int_E|K_s^{1,u,v}(e)|l(e)\lambda(e))ds\mid\mathcal{F}_t]\le C\delta^{\frac{5}{4}},\quad P\text{-a.s.},$$

where the constant C is independent of the controls $u,\ v$ and of $\delta>0$.

Proof. From $Y_s^{1,u,v}=Y_s^{u,v}-\varphi(s,X_s^{u,v})$ we have

$$\begin{array}{rcl} Y_s^{1,u,v} &=& \int_s^{t+\delta}f(r,X_r^{u,v},Y_r^{u,v},Z_r^{u,v},\int_E K_r^{u,v}(e)l(e)\lambda(de),u_r,v_r)dr-\int_s^{t+\delta}Z_r^{u,v}dB_r\\ && -\int_s^{t+\delta}\int_E K_s^{u,v}(e)\tilde{\mu}(ds,de)+\varphi(t+\delta,X_{t+\delta}^{u,v})-\varphi(s,X_s^{u,v}),\ s\in[t,t+\delta]. \end{array}$$

Then,

$$\begin{array}{rcl} Y_s^{1,u,v} & = & E[\int_s^{t+\delta} f(r, X_r^{u,v}, Y_r^{u,v}, Z_r^{u,v}, \int_E K_r^{u,v}(e)l(e)\lambda(de), u_r, v_r)dr \mid \mathcal{F}_s] \\ & & +E[\varphi(t+\delta, X_{t+\delta}^{u,v}) - \varphi(s, X_s^{u,v}) \mid \mathcal{F}_s], \quad s \in [t, t+\delta]. \end{array}$$

Therefore, from (4.6)-(i)

$$\begin{array}{rcl} |Y_s^{1,u,v}| & \leq & CE[\int_s^{t+\delta}(1+|X_r^{u,v}|+|Y_r^{u,v}|+|Z_r^{u,v}|+\int_E |K_r^{u,v}(e)||l(e)|\lambda(de))dr \mid \mathcal{F}_s] \\ & & +C\delta + CE[|X_{t+\delta}^u - X_s^u| \mid \mathcal{F}_s] \\ & \leq & C\delta^{\frac{1}{2}}(E[\int_s^{t+\delta}(1+|X_r^{u,v}|^2+|Y_r^{u,v}|^2+|Z_r^{u,v}|^2+\int_E |K_r^{u,v}(e)|^2\lambda(de))dr \mid \mathcal{F}_s])^{\frac{1}{2}} \\ & & +C\delta + C\delta^{\frac{1}{2}}(1+|X_s^{u,v}|) \\ & \leq & C\delta^{\frac{1}{2}}(1+|X_s^{u,v}|), \quad \text{P-a.s.}, \ s \in [t, t+\delta], \end{array} \tag{4.10}$$

and in particular,

$$|Y_t^{1,u,v}| \leq C(1+|x|), \ \text{P-a.s.}$$

From

$$\begin{array}{l} Z_s^{1,u,v} = Z_s^{u,v} - D\varphi(s, X_s^{u,v}).\sigma(s, X_s^{u,v}, Y_s^{u,v}, Z_s^{u,v}, u_s, v_s), \\ K_s^{1,u,v}(e) = K_s^{u,v}(e) - \varphi(s, X_s^{u,v} + h(s, X_{s-}^{u,v}, Y_{s-}^{u,v}, Z_s^{u,v}, u_s, v_s, e) + \varphi(s, X_s^{u,v}) \end{array}$$

we get

$$\begin{array}{l} |Z_s^{1,u,v}| \leq C(1+|X_s^{u,v}|+|Y_s^{u,v}|+|Z_s^{u,v}|), \\ |K_s^{1,u,v}(e)| \leq C(1+|X_s^{u,v}|+|Y_s^{u,v}|+|Z_s^{u,v}|+|K_s^{u,v}(e)|), \ \text{P-a.s.}, \ s \in [t, t+\delta]. \end{array} \tag{4.11}$$

Moreover, from $Y_s^{1,u,v} = Y_s^{u,v} - \varphi(s, X_s^{u,v})$ combined with (4.10), we get

$$|Y_s^{u,v}| \leq C(1+|X_s^{u,v}|), \ \text{P-a.s.}, \ s \in [t, t+\delta]. \tag{4.12}$$

On the other hand, from Theorem 4.2, we know $Z_s^{u,v} = q(s, X_s^{u,v}, Y_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s)$, where the function q is Lipschitz in y, z, and of linear growth. Putting $F(s,x,y,z,k,u,v) = L(s,x,y,q(s,x,y,z,u,v),k,u,v)$, (4.7) can be rewritten as follows: $s \in [t, t+\delta]$,

$$Y_s^{1,u,v} = \int_s^{t+\delta} F(r, X_r^{u,v}, Y_r^{1,u,v}, Z_r^{1,u,v}, K_r^{1,u,v}, u_r, v_r)dr - \int_s^{t+\delta} Z_r^{1,u,v}dB_r - \int_s^{t+\delta}\int_E K_s^{1,u,v}(e)\tilde{\mu}(ds,de).$$

Therefore, from (4.10), (4.11) as well as (4.6)-(i)-(iii)

$$\begin{array}{rl} & |Y_t^{1,u,v}|^2 + E[\int_t^{t+\delta}|Z_r^{1,u,v}|^2dr + \int_t^{t+\delta}\int_E |K_r^{1,u,v}(e)|^2\lambda(de)dr \mid \mathcal{F}_t] \\ = & 2E[\int_t^{t+\delta} Y_r^{1,u,v}F(r, X_r^{u,v}, Y_r^{1,u,v}, Z_r^{1,u,v}, K_r^{1,u,v}, u_r, v_r)dr \mid \mathcal{F}_t] \\ \leq & CE[\int_t^{t+\delta}|Y_r^{1,u,v}|(1+|X_r^{u,v}|^2+|Y_r^{1,u,v}|^2+|Z_r^{1,u,v}|^2+\int_E |K_r^{u,v}(e)||l(e)|\lambda(de))dr \mid \mathcal{F}_t] \\ \leq & CE[\int_t^{t+\delta}|Y_r^{1,u,v}|(1+|X_r^{u,v}|^2+|Y_r^{1,u,v}|^2+|Z_r^{u,v}|^2+\int_E |K_r^{u,v}(e)||l(e)|\lambda(de))dr \mid \mathcal{F}_t] \\ \leq & C\delta^{\frac{1}{2}}E[\int_t^{t+\delta}(1+|X_r^{u,v}|^2+|X_r^{u,v}|^3)dr \mid \mathcal{F}_t] + C\delta^{\frac{1}{2}}E[\int_t^{t+\delta}(1+|X_r^{u,v}|)|Z_r^{u,v}|^2dr \mid \mathcal{F}_t] \\ & +C\delta^{\frac{1}{2}}E[\int_t^{t+\delta}\int_E |K_r^{u,v}(e)|^2\lambda(de)dr \mid \mathcal{F}_t] \\ \leq & C\delta^{\frac{3}{2}}. \end{array} \tag{4.13}$$

Therefore,

$$\begin{array}{rl} & E[\int_t^{t+\delta}(|Y_r^{1,u,v}|+|Z_r^{1,u,v}|+\int_E |K_r^{u,v}(e)|l(e)\lambda(de))ds \mid \mathcal{F}_t] \\ \leq & C\delta^{\frac{1}{2}}E[\int_t^{t+\delta}(1+|X_r^{u,v}|)dr \mid \mathcal{F}_t] + C\delta^{\frac{1}{2}}(E[\int_t^{t+\delta}|Z_r^{1,u,v}|^2dr \mid \mathcal{F}_t])^{\frac{1}{2}} \\ & +C\delta^{\frac{1}{2}}(E[\int_t^{t+\delta}\int_E |K_r^{1,u,v}(e)|^2\lambda(de)dr \mid \mathcal{F}_t])^{\frac{1}{2}} \\ \leq & C\delta^{\frac{5}{4}}, \quad \text{P-a.s.}, \ 0 \leq \delta \leq \delta_1. \end{array}$$

□

Remark 4.4. *From (4.6)-*(iii) *we know*

$$E[(\int_t^{t+\delta}|Z_s^{u,v}|^2ds)^2 \mid \mathcal{F}_t] \leq C\delta^2, \quad P\text{-a.s.}$$

Then, from (4.6)-(i)*, (4.10), (4.11) and (4.12),*

$$E[(\int_t^{t+\delta}|Z_s^{1,u,v}|^2ds)^2 \mid \mathcal{F}_t] \leq C\delta^2, \quad P\text{-a.s.} \tag{4.14}$$

Remark 4.5. *As $(Y_s^{1,u,v}, Z_s^{1,u,v}) \in \mathcal{S}^2(t,t+\delta;\mathbb{R}) \times \mathcal{M}^2(t,t+\delta;\mathbb{R}^d)$, we have from the algebraic equation in (4.8) that also $\hat{Z}^{u,v} \in \mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$.*

Moreover, from Theorem 4.2, we know $\hat{Z}_s^{u,v} = q(s,x,Y_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s)$. Therefore, similar to Remark 4.4, we have

$$\begin{array}{ll}\text{(i)} & E[\int_t^{t+\delta} |\hat{Z}_s^{u,v}|^2 ds \mid \mathcal{F}_t] \le C\delta, \quad P\text{-a.s.} \\ \text{(ii)} & E[(\int_t^{t+\delta} |\hat{Z}_s^{u,v}|^2 ds)^2 \mid \mathcal{F}_t] \le C\delta^2, \quad P\text{-a.s.}\end{array}$$

Lemma 4.3. *For every $u \in \mathcal{U}_{t,t+\delta},\ v \in \mathcal{V}_{t,t+\delta}$, we have*

$$|Y_t^{1,u,v} - Y_t^{2,u,v}| \le C\delta^{\frac{5}{4}}, \quad P\text{-a.s.}, \quad 0 \le \delta \le \bar{\delta}_0,$$

where C is independent of the control processes u,v and of $\delta > 0$.

Proof. We set $g(s) = L(s, X_s^{u,v}, 0, Z_s^{u,v}, K_s^{1,u,v}, u_s, v_s) - L(s, x, 0, Z_s^{u,v}, K_s^{1,u,v}, u_s, v_s)$ and $\rho_0(r) = (1+|x|^2 + |Z_s^{u,v}|)(r+r^2),\ r \ge 0$. Obviously, $|g(s)| \le C\rho_0(|X_s^{u,v} - x|)$, for $s \in [t,t+\delta],\ (t,x) \in [0,T) \times \mathbb{R},\ u \in \mathcal{U}_{t,t+\delta},\ v \in \mathcal{V}_{t,t+\delta}$. Therefore, we have, from equations (4.7) and (4.8),

$$\begin{array}{rl} & |Y_t^{1,u,v} - Y_t^{2,u,v}| = |E[|Y_t^{1,u,v} - Y_t^{2,u,v}| \mid \mathcal{F}_t]| \\ = & |E[\int_t^{t+\delta} (L(s, X_s^{u,v}, Y_s^{1,u,v}, Z_s^{u,v}, K_s^{1,u,v}, u_s, v_s) - L(s,x,0,\hat{Z}_s^{u,v}, 0, u_s, v_s)) ds \mid \mathcal{F}_t]| \\ \le & CE[\int_t^{t+\delta} (\rho_0(|X_s^{u,v} - x|) + C(1 + |X_s^{u,v}| + |Y_s^{1,u,v}| + |Z_s^{u,v}|)|Y_s^{1,u,v}| \\ & +C(1+|x| + |Z_s^{u,v}| + |Z_s^{u,v} - \hat{Z}_s^{u,v}|)|Z_s^{u,v} - \hat{Z}_s^{u,v}| + \int_E |K_s^{1,u,v}(e)| l(e)\lambda(de)) ds \mid \mathcal{F}_t] \\ \le & C\delta^{\frac{5}{4}} + CE[\int_t^{t+\delta} |Z_s^{u,v} - \hat{Z}_s^{u,v}| ds \mid \mathcal{F}_t] + CE[\int_t^{t+\delta} |Z_s^{u,v}||Z_s^{u,v} - \hat{Z}_s^{u,v}| ds \mid \mathcal{F}_t] \\ & +CE[\int_t^{t+\delta} |Z_s^{u,v} - \hat{Z}_s^{u,v}|^2 ds \mid \mathcal{F}_t] + CE[\int_t^{t+\delta} \int_E |K_s^{1,u,v}(e)| l(e)\lambda(de) ds \mid \mathcal{F}_t]. \end{array} \tag{4.15}$$

Similar to Lemma 4.6 in [18], we have

$$|Z_s^{u,v} - \hat{Z}_s^{u,v}| \le C(1 + |X_s^{u,v}|)|X_s^{u,v} - x| + C|X_s^{u,v} - x|(|Y_s^{1,u,v}| + |Z_s^{u,v}|),$$

and from (4.6), we know

$$E[\int_t^{t+\delta} |Z_s^{u,v} - \hat{Z}_s^{u,v}| ds \mid \mathcal{F}_t] \le \delta^{\frac{3}{2}},\ E[\int_t^{t+\delta} |Z_s^{u,v}||Z_s^{u,v} - \hat{Z}_s^{u,v}| ds \mid \mathcal{F}_t] \le \delta^{\frac{3}{2}},\ E[\int_t^{t+\delta} |Z_s^{u,v} - \hat{Z}_s^{u,v}|^2 ds \mid \mathcal{F}_t] \le \delta^{\frac{3}{2}}.$$

Then, Lemma 4.2 allows to complete the proof. □

We now consider the following equation

$$\left\{ \begin{array}{lll} dY_s^{3,u,v} & = & -L(s,x,0,\psi(s,x,u_s,v_s), 0, u_s, v_s) ds + Z_s^{3,u,v} dB_s + \int_E K_s^{3,u,v}(e) \tilde{\mu}(ds,de),\ s \in [t,t+\delta], \\ \psi(s,x,u,v) & = & D\varphi(s,x).\sigma(s,x,\varphi(s,x),\psi(s,x,u,v),u,v),\ s \in [t,t+\delta], \\ Y_{t+\delta}^{3,u,v} & = & 0. \end{array} \right. \tag{4.16}$$

Lemma 4.4. *For every $u \in \mathcal{U}_{t,t+\delta},\ v \in \mathcal{V}_{t,t+\delta}$, we have*

$$|Y_t^{2,u,v} - Y_t^{3,u,v}| \le C\delta^{\frac{5}{4}},\ P\text{-a.s.}, \quad 0 \le \delta \le \bar{\delta}_0,$$

where C is independent of the control processes u,v and of $\delta > 0$.

Proof.

$$\begin{array}{rcl} |Y_t^{2,u,v} - Y_t^{3,u,v}| & = & |E[\int_t^{t+\delta} (L(s,x,0,\hat{Z}_s^{u,v}, 0, u_s, v_s) - L(s,x,0,\psi(s,x,u_s,v_s), 0, u_s, v_s)) ds \mid \mathcal{F}_t]| \\ & \le & CE[\int_t^{t+\delta} (1 + |x| + |\hat{Z}_s^{u,v}|)|\hat{Z}_s^{u,v} - \psi(s,x,u_s,v_s)| ds \mid \mathcal{F}_t]. \end{array}$$

From Remark 4.5 we have $\hat{Z}_s^{u,v} = q(s,x,Y_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s)$, and from Theorem 4.2 $\psi(s,x,u_s,v_s) = q(s,x,0,0,u_s,v_s)$. Hence, we obtain $|\hat{Z}_s^{u,v} - \psi(s,x,u_s,v_s)| \le C(|Y_s^{1,u,v}| + |Z_s^{1,u,v}|)$. Moreover, from (4.13), (4.14) and Remark 4.5,

$$\begin{aligned} & E[\int_t^{t+\delta} |\hat{Z}_s^{u,v}||\hat{Z}_s^{u,v} - \psi(s,x,u_s,v_s)|ds \mid \mathcal{F}_t] \le CE[\int_t^{t+\delta} |\hat{Z}_s^{u,v}|(|Y_s^{1,u,v}| + |Z_s^{1,u,v}|)ds \mid \mathcal{F}_t] \\ \le\ & C(E[\int_t^{t+\delta} |\hat{Z}_s^{u,v}|^2 ds \mid \mathcal{F}_t])^{\frac{1}{2}} (E[\int_t^{t+\delta}(|Y_s^{1,u,v}| + |Z_s^{1,u,v}|)^2 ds \mid \mathcal{F}_t])^{\frac{1}{2}} \\ \le\ & C\delta^{\frac{5}{4}}, \ \text{ P-a.s.} \end{aligned}$$

From Lemma 4.2, we have $|Y_t^{2,u,v} - Y_t^{3,u,v}| \le C\delta^{\frac{5}{4}}$.

□

Lemma 4.5. *Let $Y_0(\cdot)$ be the solution of the following ordinary differential equation combined with an algebraic equation:*

$$\left\{ \begin{array}{lll} -dY_0(s) & = & L_0(s,x,0)ds, \ \ s \in [t,t+\delta], \\ \psi(s,x,u,v) & = & D\varphi(s,x).\sigma(s,x,\varphi(s,x),\psi(s,x,u,v),u,v), \ \ s \in [t,t+\delta], \\ Y_0(t+\delta) & = & 0, \end{array} \right. \tag{4.17}$$

where the function L_0 is defined by

$$L_0(s,x,0) = \sup_{u\in U} \inf_{v\in V} L(s,x,0,\psi(s,x,u,v),0,u,v), \ \ (s,x) \in [t,t+\delta] \times \mathbb{R}.$$

Then, P-a.s., $Y_0(t) = \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} \operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}} Y_t^{3,u,v}$.

Proof. First we introduce the function

$$L_1(s,x,0,u) = \inf_{v\in V} L(s,x,0,\psi(s,x,u,v),0,u,v), \ \ (s,x,u) \in [0,T] \times \mathbb{R} \times U$$

and consider the following equation:

$$\left\{ \begin{array}{lll} -dY_s^{4,u} & = & L_1(s,x,0,u_s)ds - Z_s^{4,u}dB_s - \int_E K_s^{4,u}(e)\tilde{\mu}(ds,de), \quad s \in [t,t+\delta], \\ \psi(s,x,u,v) & = & D\varphi(s,x).\sigma(s,x,\varphi(s,x),\psi(s,x,u,v),u,v), \\ Y_{t+\delta}^{4,u} & = & 0. \end{array} \right. \tag{4.18}$$

From Lemma 2.1, for every $u \in \mathcal{U}_{t,t+\delta}$, there exists a unique solution $(Y^{4,u}, Z^{4,u}, K^{4,u})$ to (4.18). Moreover,

$$Y_t^{4,u} = \operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}} Y_t^{3,u,v}, \ \text{ P-a.s. for every } u \in \mathcal{U}_{t,t+\delta}.$$

In fact, from the definition of L_1 and Lemma 2.2 (comparison theorem), we have

$$Y_t^{4,u} \le Y_t^{3,u,v}, \ \text{ P-a.s. for any } v \in \mathcal{V}_{t,t+\delta}, \ \text{ for every } u \in \mathcal{U}_{t,t+\delta}.$$

On the other hand, there exists a measurable function $v^4 : [t,t+\delta] \times \mathbb{R} \times U \to V$ such that

$$L_1(s,x,0,u) = L(s,x,0,\psi(s,x,u,v^4(s,x,u)),0,u,v^4(s,x,u)), \ \text{ for any } s,x,u.$$

We then put $\widetilde{v}_s^4 = v^4(s,x,u_s)$, $s \in [t,t+\delta]$, and we observe that $\widetilde{v}^4 \in \mathcal{V}_{t,t+\delta}$ ($\widetilde{v}^4$ depends on $u \in \mathcal{U}_{t,t+\delta}$) and

$$L_1(s,x,0,u_s) = L(s,x,0,\psi(s,x,u_s,\widetilde{v}_s^4),0,u_s,\widetilde{v}_s^4), \ \ s \in [t,t+\delta].$$

Consequently, from the uniqueness of the solution of (4.18) it follows that $(Y^{4,u}, Z^{4,u}) = (Y^{3,u,\widetilde{v}^4}, Z^{3,u,\widetilde{v}^4})$, and in particular, $Y_t^{4,u} = Y_t^{3,u,\widetilde{v}^4}$, P-a.s. This proves that $Y_t^{4,u} = \operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}} Y_t^{3,u,v}$, P-a.s., for every $u \in \mathcal{U}_{t,t+\delta}$. Finally, since $L_0(s,x,0) = \sup_{u\in U} L_1(s,x,0,u)$, by a similar proof we show that

$$Y_0(t) = \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} Y_t^{4,u} = \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} \operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}} Y_t^{3,u,v}, \text{P-a.s.}$$

□

We are now able to finish the proof of Theorem 4.1.

Proof. (1) First we will prove that W is a viscosity supersolution. Obviously, $W(T,x)=\Phi(x),\ x\in\mathbb{R}$. We suppose that $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R})$ satisfying the monotonicity condition (**H2.3**)-(ii)' if σ depends on z from (**H4.1**) and $(t,x)\in[0,T)\times\mathbb{R}$ are such that $W-\varphi$ attains its minimum at (t,x). Since W is continuous and of at most linear growth, we can replace the condition of a local minimum by that of a global one in the definition of the viscosity supersolution. Moreover, without loss of generality we may assume that $\varphi(t,x)=W(t,x)$. Due to the DPP (Theorem 3.1), we have

$$\varphi(t,x)=W(t,x)=\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u,\beta(u)}_{t+\delta})],\ 0\le\delta\le\delta_0,$$

where $\widetilde{X}^{t,x;u,\beta(u)}$ is defined by FBSDE (3.11). From $\varphi(s,y)\le W(s,y),\ (s,y)\in[0,T)\times\mathbb{R}$, and the monotonicity property of $G^{t,x;u,\beta(u)}_{t,t+\delta}[\cdot]$ (see Theorem 3.3 in [19]) we obtain

$$\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}G^{t,x;u,\beta(u)}_{t,t+\delta}[\varphi(t+\delta,X^{u,\beta(u)}_{t+\delta})]-\varphi(t,x)\le 0. \tag{4.19}$$

According to the definition of the backward stochastic semigroup for fully coupled FBSDE with jumps, we have

$$G^{t,x;u,v}_{s,t+\delta}[\varphi(t+\delta,X^{u,v}_{t+\delta})]=Y^{u,v}_s,\quad s\in[t,t+\delta].$$

Moreover, $Y^{1,u,v}_s=Y^{u,v}_s-\varphi(s,X^{u,v}_s)$; therefore, we have

$$\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y^{1,u,\beta(u)}_t\le 0,\ \text{P-a.s.}$$

Thus, from the Lemmas 4.3 and 4.4 we get $\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y^{3,u,\beta(u)}_t\le C\delta^{\frac54},$ P-a.s. Consequently, since $\operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}}Y^{3,u,v}_t\le Y^{3,u,\beta(u)}_t,\ \beta\in\mathcal{B}_{t,t+\delta}$, we get

$$\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}\operatorname*{essinf}_{v\in\mathcal{V}_{t,t+\delta}}Y^{3,u,v}_t\le\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y^{3,u,\beta(u)}_t\le C\delta^{\frac54},\ \text{P-a.s.}$$

Thus, by Lemma 4.5, $Y_0(t)\le C\delta^{\frac54}$, P-a.s., where $Y_0(\cdot)$ is the unique solution of (4.17). Consequently,

$$C\delta^{\frac14}\ge\frac1\delta Y_0(t)=\frac1\delta\int_t^{t+\delta}L_0(s,x,0)ds,\ \delta>0.$$

Thanks to the continuity of $s\mapsto L_0(s,x,0)$ it follows that

$$\sup_{u\in U}\inf_{v\in V}L(t,x,0,\psi(t,x,u,v),0,u,v)=L_0(t,x,0)\le 0,$$

where $\psi(t,x,u,v)=D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x,u,v),u,v)$. From the definition of L we see that W is a viscosity supersolution of (4.2).

(2) Now we prove W is a viscosity subsolution. For this we suppose that $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R})$ satisfying the monotonicity condition (**H2.3**)-(ii)' if σ depends on z from (**H4.1**) and $(t,x)\in[0,T)\times\mathbb{R}$ are such that $W-\varphi$ attains its maximum at (t,x). Without loss of generality we may also suppose that $\varphi(t,x)=W(t,x)$. We must prove that

$$\sup_{u\in\mathcal{U}}\inf_{v\in V}L(t,x,0,\psi(t,x,u,v),0,u,v)=L_0(t,x,0)\ge 0,$$

where $\psi(t,x,u,v)=D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x,u,v),u,v)$. Let us suppose that this is not true. Then there exists some $\theta>0$ such that

$$L_0(t,x,0)=\sup_{u\in\mathcal{U}}\inf_{v\in V}L(t,x,0,\psi(t,x,u,v),0,u,v)\le-\theta<0. \tag{4.20}$$

and we can find a measurable function $\gamma : U \to V$ such that

$$L(t,x,0,\psi(t,x,u,\gamma(u)),0,u,\gamma(u)) \le -\frac{3}{4}\theta, \text{ for all } u \in U.$$

Moreover, since $L(\cdot,x,0,\psi(\cdot,x,\cdot,\cdot),0,\cdot,\cdot)$ is uniformly continuous on $[0,T]\times U\times V$, there exists some $T-t \ge R > 0$ such that

$$L(s,x,0,\psi(s,x,u,\gamma(u)),0,u,\gamma(u)) \le -\frac{1}{2}\theta, \text{ for all } u \in U \text{ and } |s-t| \le R. \tag{4.21}$$

On the other hand, due to the DPP (see Theorem 3.1),

$$\varphi(t,x) = W(t,x) = \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x,u,\beta(u)}_{t+\delta})],\ 0\le\delta\le\delta_0,$$

where $\widetilde{X}^{t,x;u,\beta(u)}$ is defined by FBSDE (3.11). And from $W \le \varphi$ and the monotonicity property of $G^{t,x;u,\beta(u)}_{t,t+\delta}[\cdot]$ (see Theorem 3.3 in [19]) we obtain

$$\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[\varphi(t+\delta,X^{u,\beta(u)}_{t+\delta})] - \varphi(t,x) \ge 0, \text{ P-a.s.,}$$

Then, similar to (1), from the definition of backward semigroup, we have

$$\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} Y^{1,u,\beta(u)}_t \ge 0, \text{ P-a.s.,}$$

and, in particular, $\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} Y^{1,u,\gamma(u)}_t \ge 0$, P-a.s. Here, by putting $\gamma_s(u)(\omega)=\gamma(u_s(\omega)),\ (s,\omega)\in[t,T]\times\Omega$, we identify γ as an element of $\mathcal{B}_{t,t+\delta}$. Given any $\varepsilon>0$ we can choose $u^\varepsilon\in\mathcal{U}_{t,t+\delta}$ such that $Y^{1,u^\varepsilon,\gamma(u^\varepsilon)}_t \ge -\varepsilon\delta$. From the Lemmas 4.3 and 4.4 we have

$$Y^{3,u^\varepsilon,\gamma(u^\varepsilon)}_t \ge -C\delta^{\frac{5}{4}} - \varepsilon\delta, \text{ P-a.s.} \tag{4.22}$$

Moreover, from (4.16)

$$Y^{3,u^\varepsilon,\gamma(u^\varepsilon)}_t = E[\int_t^{t+\delta} L(s,x,0,\psi(s,x,u^\varepsilon_s,\gamma(u^\varepsilon_s)),0,u^\varepsilon_s,\gamma(u^\varepsilon_s))ds|\mathcal{F}_t],$$

and we get from (4.21)

$$Y^{3,u^\varepsilon,\gamma(u^\varepsilon)}_t \le E[\textstyle\int_t^{t+\delta}|L(s,x,0,\psi(s,x,u^\varepsilon_s,\gamma(u^\varepsilon_s)),0,u^\varepsilon_s,\gamma(u^\varepsilon_s))|ds|\mathcal{F}_t] \le -\frac{1}{2}\theta\delta, \text{ P-a.s.} \tag{4.23}$$

From (4.22) and (4.23), $-C\delta^{\frac{1}{4}}-\varepsilon \le -\frac{1}{2}\theta$, P-a.s. Letting $\delta\downarrow 0$, and then $\varepsilon\downarrow 0$, we get that $\theta\le 0$, which yields a contradiction. Therefore,

$$\sup_{u\in U}\inf_{v\in V} L(t,x,0,\psi(t,x,u,v),0,u,v) = L_0(t,x,0) \ge 0,$$

and from the definition of L we see that W is a viscosity supersolution of (4.2). Finally, from the above two steps, we derive that W is a viscosity solution of (4.2).

□

5 Viscosity solution of Isaacs' equation: Uniqueness Theorem

In this section, we will state the uniqueness of the viscosity solution of Isaacs' equation (4.2), in which $\sigma,\ h$ do not depend on $y,\ z,\ k$, i.e.

$$\begin{cases} \frac{\partial}{\partial t}W(t,x) + H^-(t,x,W,DW,D^2W) = 0, & (t,x)\in[0,T)\times\mathbb{R}, \\ W(T,x) = \Phi(x), \quad x\in\mathbb{R}, \end{cases} \tag{5.1}$$

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) + H^+(t,x,U,DU,D^2U) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ U(T,x) = \Phi(x), & x \in \mathbb{R}, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} H^-(t,x,W,DW,D^2W) &= \sup_{u\in U}\inf_{v\in V} H(t,x,W,DW,D^2W,u,v), \\ H^+(t,x,U,DU,D^2U) &= \inf_{v\in V}\sup_{u\in U} H(t,x,U,DU,D^2U,u,v) \end{aligned}$$

and

$$\begin{aligned} &H(t,x,W,DW,D^2W) \\ &= \tfrac{1}{2}tr(\sigma\sigma^T(t,x,u,v)D^2W(t,x)) + DW(t,x).b(t,x,W(t,x),DW(t,x).\sigma(t,x,u,v),u,v) \\ &\quad + \textstyle\int_E [W(t,x+h(t,x,u,v,e)) - W(t,x) - DW(t,x).h(t,x,u,v,e)]\lambda(de) \\ &\quad + f(t,x,W(t,x),DW(t,x).\sigma(t,x,u,v),\textstyle\int_E [W(t,x+h(t,x,u,v,e)) - W(t,x)]l(e)\lambda(de),u,v), \end{aligned}$$

where $t \in [0,T],\ x \in \mathbb{R}$.
Set

$$\begin{aligned} \Theta = \{\varphi \in C([0,T]\times\mathbb{R}) : \exists \tilde{A} > 0 \text{ such that } \lim_{|x|\to\infty} \varphi(t,x)\exp\{-\tilde{A}[\log((|x|^2+1)^{\frac{1}{2}})]^2\} = 0, \\ \text{uniformly in } t \in [0,T]\}. \end{aligned}$$

We will prove the uniqueness for equation (5.1) in Θ. The growth condition in Θ is weaker than the polynomial growth but more restrictive than the exponential growth. Barles, Buckdahn and Pardoux [2], Barles, Imbert [3] introduced this growth condition (which is optimal for the uniqueness and can not be weaken in general) to prove the uniqueness of the viscosity solution of an integral-partial differential equation associated with a decoupled FBSDE with jumps but without controls. Next, by applying the method developed in [2] and [3], we get the uniqueness of the viscosity solution of (5.1) in Θ. The proof for (5.2) is similar. On the other hand, since σ does not depend on z, we don't need the assumption (**H4.1**), that is, the test function φ in Definitions 4.1 and 4.2 does not need to satisfy (**H2.3**)-(ii)' now. First we present two auxiliary lemmas.

Lemma 5.1. *Let $w_1 \in \Theta$ be a viscosity subsolution and $w_2 \in \Theta$ be a viscosity supersolution of equation (5.1). Then the function $w := w_1 - w_2$ is a viscosity subsolution of the equation*

$$\begin{cases} \frac{\partial}{\partial t}w(t,x) + \sup\limits_{u\in U, v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t,x,u,v)D^2w) + Dw.b(t,x,w_1(t,x),0,u,v) + B^{u,v}w(t,x) + \tilde{K}|w(t,x)| \\ +\tilde{K}|Dw(t,x).\sigma(t,x,u,v)| + \tilde{K}(C^{u,v}w(t,x))^+\} = 0 \\ w(T,x) = 0, \quad x \in \mathbb{R}, \end{cases} \tag{5.3}$$

where $\tilde{K}$ is a constant depending on the Lipschitz constants of $b,\ \sigma,\ h,\ f$, which is uniformly in (t,u,v).

Proof. With the help of Lemma 7 in Nie [23], combined with Lemma 3.7 in [2], we can obtain the result.

Let $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R})$, and let $(t_0,x_0) \in (0,T)\times\mathbb{R}$ be a maximum point of $w-\varphi$ and $w(t_0,x_0) = \varphi(t_0,x_0)$. Without loss of generality assume that (t_0,x_0) is a strict global maximum point of $w-\varphi$, otherwise, we can modify φ outside a small neighborhood of (t_0,x_0) if necessary. Also, the Lipschitz property of w_1 and w_2 allows to assume that $D\varphi$ is uniformly bounded: $|D\varphi| \leq K_{w_1,w_2}$. For a given $\varepsilon > 0$, define

$$\psi_\varepsilon(t,x,y) = w_1(t,x) - w_2(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \varphi(t,x).$$

From Proposition 3.7 in [7], we conclude that there exists a sequence $(t_\varepsilon,x_\varepsilon,y_\varepsilon)$ such that

(i) $(t_\varepsilon,x_\varepsilon,y_\varepsilon)$ is a global maximum point of ψ_ε in $([0,T]\times\bar{B}_R)^2$, where B_R is a ball with a large radius R;

(ii) $(t_\varepsilon,x_\varepsilon,y_\varepsilon) \to (t_0,x_0,x_0)$, as $\varepsilon \to 0$;

(iii) $\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}$ is bounded and tends to 0, when $\varepsilon \to 0$.

Moreover, since (t_0,x_0) is a strict global maximum point of $w_1-w_2-\varphi$ and $\psi_\varepsilon(t_0,x_0,x_0)\leq\psi_\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon)$, we have

$$\begin{array}{rcl}0&=&w_1(t_0,x_0)-w_2(t_0,x_0)-\varphi(t_0,x_0)=\psi_\varepsilon(t_0,x_0,x_0)\leq\psi_\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon)\\&=&w(t_\varepsilon,x_\varepsilon)-\varphi(t_\varepsilon,x_\varepsilon)+w_2(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)-\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}\\&\leq&w_2(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)-\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2},\end{array}$$

from which we know $\frac{|x_\varepsilon-y_\varepsilon|}{\varepsilon^2}\leq|\frac{w_2(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)}{x_\varepsilon-y_\varepsilon}|\leq K_{w_2}$.

Furthermore, from Theorem 8.3 in [7], for any $\alpha>0$, there exists $(X^\alpha,Y^\alpha)\in\mathcal{S}^d\times\mathcal{S}^d,\ c^\alpha\in\mathbb{R}$ such that

$$\begin{array}{l}(c^\alpha+\frac{\partial}{\partial t}\varphi(t_\varepsilon,x_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon),X^\alpha)\in\bar{\mathcal{P}}^{2,+}w_1(t_\varepsilon,x_\varepsilon),\\(c^\alpha,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2},Y^\alpha)\in\bar{\mathcal{P}}^{2,-}w_2(t_\varepsilon,x_\varepsilon),\end{array}$$

and

$$\left(\begin{array}{cc}X^\alpha&0\\0&-Y^\alpha\end{array}\right)\leq A+\delta A^2,$$

where $A=\left(\begin{array}{cc}D^2\varphi(t_\varepsilon,x_\varepsilon)+\frac{2}{\varepsilon^2}&-\frac{2}{\varepsilon^2}\\-\frac{2}{\varepsilon^2}&\frac{2}{\varepsilon^2}\end{array}\right)$.

Since w_1 and w_2 are sub- and supersolution of (5.1), respectively, from the definitions of the viscosity solution, we have, for the sufficiently small δ,

$$\begin{array}{l}c^\alpha+\frac{\partial\varphi}{\partial t}(t_\varepsilon,x_\varepsilon)+\sup\limits_{u\in U}\inf\limits_{v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,x_\varepsilon,u,v)X^\alpha)\\+\langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon)\rangle\\+\int_{E_\delta}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)+\int_{E_\delta}(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon)-D\varphi(t_\varepsilon,x_\varepsilon)h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de)\\+\int_{E_\delta^c}(w_1(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-w_1(t_\varepsilon,x_\varepsilon)-(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de)\\+f(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),B_1^\delta,u,v)\}\geq0,\end{array}\tag{5.4}$$

where

$$\begin{array}{rcl}B_1^\delta&=&\int_{E_\delta}(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,x_\varepsilon,u,v,e)+\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)|^2}{\varepsilon^2})l(e)\lambda(de)\\&&+\int_{E_\delta}(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon))l(e)\lambda(de)\\&&+\int_{E_\delta^c}(w_1(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-w_1(t_\varepsilon,x_\varepsilon))l(e)\lambda(de),\end{array}$$

and

$$\begin{array}{l}c^\alpha+\sup\limits_{u\in U}\inf\limits_{v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,y_\varepsilon,u,v)Y^\alpha)+\langle b(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\rangle\\-\int_{E_\delta}\frac{|h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)+\int_{E_\delta^c}(w_2(t_\varepsilon,y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))-w_2(t_\varepsilon,y_\varepsilon)-\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,y_\varepsilon,u,v,e))\lambda(de)\\+f(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),B_2^\delta,u,v)\}\leq0,\end{array}\tag{5.5}$$

where

$$\begin{array}{rcl}B_2^\delta&=&\int_{E_\delta}(-\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,y_\varepsilon,u,v,e)-\frac{|h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2})l(e)\lambda(de)\\&&+\int_{E_\delta^c}(w_2(t_\varepsilon,y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))-w_2(t_\varepsilon,y_\varepsilon))l(e)\lambda(de),\end{array}$$

Set

$$\begin{array}{l}I_{1,u,v}^{\varepsilon,\alpha}:=\frac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,x_\varepsilon,u,v)X^\alpha)-\frac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,y_\varepsilon,u,v)Y^\alpha),\\I_{2,u,v}^{\varepsilon,\alpha}:=\langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon)\rangle\\\qquad-\langle b(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\rangle,\\I_{3,u,v}^{\varepsilon,\alpha,\delta}:=\\\int_{E_\delta}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)+\int_{E_\delta}(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon)-D\varphi(t_\varepsilon,x_\varepsilon)h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de)\\+\int_{E_\delta^c}(w_1(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-w_1(t_\varepsilon,x_\varepsilon)-(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de)\\+\int_{E_\delta}\frac{|h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)-\int_{E_\delta^c}(w_2(t_\varepsilon,y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))-w_2(t_\varepsilon,y_\varepsilon)-\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,y_\varepsilon,u,v,e))\lambda(de),\\I_{4,u,v}^{\varepsilon,\alpha,\delta}:=f(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),B_1^\delta,u,v)\\\qquad-f(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),B_2^\delta,u,v),\end{array}$$

Then, from (5.4) and (5.5), we know

$$\frac{\partial}{\partial t}\varphi(t_\varepsilon,x_\varepsilon)+\sup_{u\in U,v\in V}\{I^{\varepsilon,\alpha}_{1,u,v}+I^{\varepsilon,\alpha}_{2,u,v}+I^{\varepsilon,\alpha,\delta}_{3,u,v}+I^{\varepsilon,\alpha,\delta}_{4,u,v}\}\geq 0. \tag{5.6}$$

For any $u\in U, v\in V$, we want to prove the following result:

$$\begin{array}{rl} & \lim\limits_{\varepsilon\to 0}\lim\limits_{\alpha\to 0}\lim\limits_{\delta\to 0}\{I^{\varepsilon,\alpha}_{1,u,v}+I^{\varepsilon,\alpha}_{2,u,v}+I^{\varepsilon,\alpha,\delta}_{3,u,v}+I^{\varepsilon,\alpha,\delta}_{4,u,v}\} \\ \leq & \frac{1}{2}tr(\sigma\sigma^T(t_0,x_0,u,v)D^2\varphi(t_0,x_0))+\langle b(t_0,x_0,w_1(t_0,x_0),0,u,v),D\varphi(t_0,x_0)\rangle+B^{u,v}\varphi(t_0,x_0) \\ & +\tilde{K}\{|\varphi(t_0,x_0)|+|D\varphi(t_0,x_0)|\cdot|\sigma(t_0,x_0,u,v)|+(C^{u,v}\varphi(t_0,x_0))^+\}. \end{array}$$

Similar to the proof of Lemma 7 in Nie [23], we obtain

$$I^{\varepsilon,\alpha}_{1,u,v}=\tfrac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,x_\varepsilon,u,v)X^\alpha)-\tfrac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,y_\varepsilon,u,v)Y^\alpha)\leq\tfrac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,x_\varepsilon,u,v)D^2\varphi(t_\varepsilon,x_\varepsilon))+K_\sigma^2\tfrac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}.$$

For $I^{\varepsilon,\alpha}_{2,u,v}$, we have

$$\begin{array}{rcl} I^{\varepsilon,\alpha}_{2,u,v} & = & \langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon)\rangle \\ & & -\langle b(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),u,v),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\rangle \\ & = & \langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,y_\varepsilon),0,u,v),D\varphi(t_\varepsilon,x_\varepsilon)\rangle+\langle\Delta b_1,D\varphi(t_\varepsilon,x_\varepsilon)\rangle+\langle\Delta b_2,\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\rangle, \end{array}$$

where

$$\begin{array}{rcl} \Delta b_1 & = & b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),u,v)-b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),0,u,v) \\ & \leq & K_b|\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon)|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|, \\ \Delta b_2 & = & b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),u,v) \\ & & -b(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),u,v) \\ & \leq & K_b\{|x_\varepsilon-y_\varepsilon|+|w_1(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)|+|D\varphi(t_\varepsilon,x_\varepsilon)|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|\}+K_bK_\sigma\frac{2|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}, \end{array}$$

therefore,

$$\begin{array}{rcl} I^{\varepsilon,\alpha}_{2,u,v} & \leq & \langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),0,u,v),D\varphi(t_\varepsilon,x_\varepsilon)\rangle \\ & & +K_b|\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon)|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|\cdot|D\varphi(t_\varepsilon,x_\varepsilon)|+K_bK_\sigma\frac{4|x_\varepsilon-y_\varepsilon|^3}{\varepsilon^4} \\ & & +K_b\frac{2|x_\varepsilon-y_\varepsilon|}{\varepsilon^2}\{|x_\varepsilon-y_\varepsilon|+|w_1(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)|+|D\varphi(t_\varepsilon,x_\varepsilon)|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|\}, \end{array}$$

Similar to the proof of Lemma 3.7 in [2], we estimate the differences of the integral-differential terms. From the fact that $(t_\varepsilon,x_\varepsilon,y_\varepsilon)$ is a global maximum point of ψ_ε in $\bar{B}_{\frac{R}{2}}$, we know

$$\psi_\varepsilon(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e),y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))\leq\psi_\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon),$$

hence,

$$\begin{array}{l} [w_1(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-w_1(t_\varepsilon,x_\varepsilon)]-[w_2(t_\varepsilon,y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))-w_2(t_\varepsilon,y_\varepsilon)] \\ -\langle\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2},h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)\rangle-\frac{1}{\varepsilon^2}|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2 \\ \leq\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon), \end{array}$$

furthermore,

$$\begin{array}{l} \int_{E_\delta^c}(w_1(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-w_1(t_\varepsilon,x_\varepsilon)-\langle\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon),h(t_\varepsilon,x_\varepsilon,u,v,e)\rangle)\lambda(de) \\ -\int_{E_\delta^c}(w_2(t_\varepsilon,y_\varepsilon+h(t_\varepsilon,y_\varepsilon,u,v,e))-w_2(t_\varepsilon,y_\varepsilon)-\langle\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2},h(t_\varepsilon,y_\varepsilon,u,v,e)\rangle)\lambda(de) \\ \leq\int_{E_\delta^c}(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon)-\langle D\varphi(t_\varepsilon,x_\varepsilon),h(t_\varepsilon,x_\varepsilon,u,v,e)\rangle)\lambda(de) \\ +\int_{E_\delta^c}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de). \end{array}$$

From these estimates,

$$\begin{array}{rcl} I^{\varepsilon,\alpha,\delta}_{3,u,v} & \leq & \int_{E_\delta}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)+\int_{E_\delta}\frac{|h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)+\int_{E_\delta^c}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de) \\ & & +\int_E(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon)-D\varphi(t_\varepsilon,x_\varepsilon)h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de), \end{array}$$

as well as

$$\begin{array}{rcl} B_1^\delta - B_2^\delta & \leq & \int_{E_\delta}(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,x_\varepsilon,u,v,e)+\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)|^2}{\varepsilon^2})l(e)\lambda(de)\\ && +\int_{E_\delta}(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}h(t_\varepsilon,y_\varepsilon,u,v,e)+\frac{|h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2})l(e)\lambda(de)\\ && +\int_E(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon))l(e)\lambda(de)\\ && +\int_{E_\delta^c}\langle\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2},h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)\rangle l(e)\lambda(de)\\ && +\int_{E_\delta^c}\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}l(e)\lambda(de). \end{array}$$

Therefore,

$$\begin{array}{rcl} I_{4,u,v}^{\varepsilon,\alpha,\delta} & \leq & |f(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),(\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}+D\varphi(t_\varepsilon,x_\varepsilon))\sigma(t_\varepsilon,x_\varepsilon,u,v),B_1^\delta,u,v)\\ && -f(t_\varepsilon,y_\varepsilon,w_2(t_\varepsilon,y_\varepsilon),\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\sigma(t_\varepsilon,y_\varepsilon,u,v),B_2^\delta,u,v)|\\ & \leq & K_f|x_\varepsilon-y_\varepsilon|+K_f|w_1(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)|+K_f|D\varphi(t_\varepsilon,x_\varepsilon))|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|+K_fK_\sigma\frac{2|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}\\ && +K_f(B_1^\delta-B_2^\delta)^+. \end{array}$$

Since $\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\leq K_{w_2}$, and $D\varphi(t_\varepsilon,x_\varepsilon)\leq K_{w_1,w_2}$, let $\delta\to 0$ with keeping ε,α fixed, we get

$$\begin{array}{rl} & I_{1,u,v}^{\varepsilon,\alpha}+I_{2,u,v}^{\varepsilon,\alpha}+I_{3,u,v}^{\varepsilon,\alpha,\delta}+I_{4,u,v}^{\varepsilon,\alpha,\delta}\\ \leq & \frac{1}{2}tr(\sigma\sigma^T(t_\varepsilon,x_\varepsilon,u,v)D^2\varphi(t_\varepsilon,x_\varepsilon))+K_\sigma^2\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}+\langle b(t_\varepsilon,x_\varepsilon,w_1(t_\varepsilon,x_\varepsilon),0,u,v),D\varphi(t_\varepsilon,x_\varepsilon)\rangle\\ & +\tilde{K}\{|D\varphi(t_\varepsilon,x_\varepsilon))|\cdot|\sigma(t_\varepsilon,x_\varepsilon,u,v)|+|w_1(t_\varepsilon,x_\varepsilon)-w_2(t_\varepsilon,y_\varepsilon)|\}+\tilde{K}\{|x_\varepsilon-y_\varepsilon|+|x_\varepsilon-y_\varepsilon|^2\}\\ & +\int_E\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}\lambda(de)\\ & +\int_E(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon)-D\varphi(t_\varepsilon,x_\varepsilon)h(t_\varepsilon,x_\varepsilon,u,v,e))\lambda(de)\\ & +K_f(\int_E(\varphi(t_\varepsilon,x_\varepsilon+h(t_\varepsilon,x_\varepsilon,u,v,e))-\varphi(t_\varepsilon,x_\varepsilon))l(e)\lambda(de)\\ & +\int_E\langle\frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2},h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)\rangle l(e)\lambda(de)\\ & +\int_E\frac{|h(t_\varepsilon,x_\varepsilon,u,v,e)-h(t_\varepsilon,y_\varepsilon,u,v,e)|^2}{\varepsilon^2}l(e)\lambda(de))^+. \end{array}$$

Finally, we let $\alpha\to 0,\ \varepsilon\to 0$, from (ii), (iii), we get

$$\begin{array}{rcl} 0 & \leq & \lim\limits_{\varepsilon\to 0}\lim\limits_{\alpha\to 0}\lim\limits_{\delta\to 0}\{\frac{\partial}{\partial t}\varphi(t_\varepsilon,x_\varepsilon)+\sup\limits_{u\in U,v\in V}\{I_{1,u,v}^{\varepsilon,\alpha}+I_{2,u,v}^{\varepsilon,\alpha}+I_{3,u,v}^{\varepsilon,\alpha,\delta}+I_{4,u,v}^{\varepsilon,\alpha,\delta}\}\}\\ & \leq & \frac{\partial}{\partial t}\varphi(t_0,x_0)+\sup\limits_{u\in U,v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t_0,x_0,u,v)D^2\varphi(t_0,x_0))+\langle b(t_0,x_0,w_1(t_0,x_0),0,u,v),D\varphi(t_0,x_0)\rangle\\ && +\int_E(\varphi(t_0,x_0+h(t_0,x_0,u,v,e))-\varphi(t_0,x_0)-D\varphi(t_0,x_0)h(t_0,x_0,u,v,e))\lambda(de)\\ && +\tilde{K}(|w_1(t_0,x_0)-w_2(t_0,x_0)|+|D\varphi(t_0,x_0)|\cdot|\sigma(t_0,x_0,u,v)|\\ && +(\int_E(\varphi(t_0,x_0+h(t_0,x_0,u,v,e))-\varphi(t_0,x_0))l(e)\lambda(de))^+)\}\\ & = & \frac{\partial}{\partial t}\varphi(t_0,x_0)+\sup\limits_{u\in U,v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t_0,x_0,u,v)D^2\varphi(t_0,x_0))+\langle b(t_0,x_0,w_1(t_0,x_0),0,u,v),D\varphi(t_0,x_0)\rangle\\ && +B^{u,v}\varphi(t_0,x_0)+\tilde{K}|\varphi(t_0,x_0)|+\tilde{K}|D\varphi(t_0,x_0)|\cdot|\sigma(t_0,x_0,u,v)|+\tilde{K}(C^{u,v}\varphi(t_0,x_0))^+\}. \end{array}$$

Therefore, w is a viscosity subsolution of (5.3).

□

Following [2, 6], we have

Lemma 5.2. *For any $\tilde{A}>0$, there exists $C_1>0$ such that the function $\chi(t,x)=\exp[(C_1(T-t)+\tilde{A})\psi(x)]$, with $\psi(x)=[\log((|x|^2+1)^{\frac{1}{2}})+1]^2,\ x\in\mathbb{R}$, satisfies*

$$\begin{array}{l} \frac{\partial}{\partial t}\chi(t,x)+\sup\limits_{u\in U,v\in V}\{\frac{1}{2}tr(\sigma\sigma^T(t,x,u,v)D^2\chi(t,x))+D\chi(t,x).b(t,x,w_1(t,x),0,u,v)+B^{u,v}\chi(t,x)\\ +\tilde{K}|\chi(t,x)|+\tilde{K}|D\chi(t,x).\sigma(t,x,u,v)|+\tilde{K}(C^{u,v}\chi(t,x))^+\}<0,\ \textit{in } [t_1,T]\times\mathbb{R}, \end{array}\tag{5.7}$$

where $t_1=T-\frac{\tilde{A}}{C_1}$.

Proof. By direct calculus we deduce

$$|D\psi(x)|\leq\frac{2[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}},\quad |D^2\psi(x)|\leq\frac{C(1+[\psi(x)]^{\frac{1}{2}})}{|x|^2+1},\ x\in\mathbb{R}.$$

Therefore, if $t \in [t_1, T]$,

$$|D\chi(t,x)| \le (C_1(T-t)+\tilde{A})\chi(t,x)|D\psi(x)| \le C\chi(t,x)\tfrac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}},$$

and

$$|D^2\chi(t,x)| \le C\chi(t,x)\frac{\psi(x)}{|x|^2+1}.$$

Notice that the above estimates do not depend on C_1 because of the definition of t_1. Then, from γ is bounded and ψ is Lipschitz continuous in $\mathbb{R}$, by a long but straight-forward calculus, we get

$$\chi(t,x+h(t,x,u,v,e)) - \chi(t,x) - D\chi(t,x).h(t,x,u,v,e) \le C\chi(t,x)\frac{\psi(x)}{|x|^2+1}|h(t,x,u,v,e)|^2,$$

and

$$\chi(t,x+h(t,x,u,v,e)) - \chi(t,x) \le C\chi(t,x)\frac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}}|h(t,x,u,v,e)|.$$

Therefore, we have

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\chi(t,x) + \sup_{u\in U, v\in V}\{\tfrac{1}{2}tr(\sigma\sigma^T(t,x,u,v)D^2\chi(t,x)) + D\chi.b(t,x,w_1(t,x),0,u,v) + B^{u,v}\chi(t,x)\\
&+\tilde{K}|\chi(t,x)| + \tilde{K}|D\chi(t,x).\sigma(t,x,u,v)| + \tilde{K}(C^{u,v}\chi(t,x))^+\}\\
&\le -\chi(t,x)\{C_1\psi(x) - C\psi(x) - C[\psi(x)]^{\frac{1}{2}} - C\tfrac{\psi(x)}{|x|^2+1} - \tilde{K} - C\tilde{K}[\psi(x)]^{\frac{1}{2}} - C\tilde{K}\tfrac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}}\}\\
&\le -\chi(t,x)\{C_1 - [C+\tilde{K}]\}\psi(x) < 0, \text{ if } C_1 > C+\tilde{K} \text{ large enough.}
\end{aligned}$$

□

Theorem 5.1. *Let w_1 (resp., w_2) $\in \Theta$ be a viscosity subsolution (resp. supersolution) of equation (5.1). Then, if w_1 (resp., w_2) is Lipschitz in x, uniformly in t, we have*

$$w_1(t,x) \le w_2(t,x), \text{ for all } (t,x) \in [0,T]\times\mathbb{R}. \tag{5.8}$$

Proof. First we consider the case when w_1 and w_2 are bounded. Set $u := w_1 - w_2$. Theorem 4.1 in [3] proves a comparison principle for bounded sub- and supersolutions of Hamilton-Jacobi-Bellman equations with nonlocal term of type (5.3). From Lemma 5.1, we know that u is a viscosity subsolution of equation (5.3). On the other hand, clearly, $\tilde{u} = 0$ is a viscosity solution, hence it is also a viscosity supersolution of equation (5.3). Thus, Theorem 4.1 in [3] implies that $w_1 - w_2 = w \le \tilde{w} = 0$, i.e., $w_1 \le w_2$ on $[0,T]\times\mathbb{R}$. Finally, if w_1, w_2 are viscosity solutions of (5.3), they are both viscosity sub- and supersolution; from the just proved comparison result we get $w_1 = w_2$.

However, under our standard assumptions, the lower value function W defined by (3.5) is not necessarily bounded, so we still need to prove the case $w_1, w_2 \in \Theta$. Set $w := w_1 - w_2$. Then, for some $\tilde{A} > 0$,

$$\lim_{|x|\to\infty} w(t,x)\exp\{-\tilde{A}[\log((|x|^2+1)^{\frac{1}{2}})]^2\} = 0,$$

uniformly with respect to $t \in [0,T]$. Accordingly, for any $\alpha > 0$, $w(t,x) - \alpha\chi(t,x)$ is bounded from above in $[t_1,T]\times\mathbb{R}$, and that

$$M := \max_{[t_1,T]\times\mathbb{R}}(w-\alpha\chi)(t,x)e^{-\tilde{K}(T-t)}$$

is achieved at some point $(t_0,x_0) \in [t_1,T]\times\mathbb{R}$ (depending on α).

Now we consider the following two cases.

(i) We assume that: $w(t_0,x_0) \le 0$, for any $\alpha > 0$. Then, $M \le 0$ and $w_1(t,x) - w_2(t,x) \le \alpha\chi(t,x)$ in $[t_1,T]\times\mathbb{R}$. Consequently, letting $\alpha \to 0$, we get

$$w_1(t,x) \le w_2(t,x), \text{ for all } (t,x) \in [t_1,T]\times\mathbb{R}.$$

(ii) Suppose that there exists some $\alpha > 0$ such that $w(t_0, x_0) > 0$. Notice that $w(t,x) - \alpha\chi(t,x) \leq (w(t_0,x_0) - \alpha\chi(t_0,x_0))e^{-\tilde{K}(t-t_0)}$ in $[t_1, T] \times \mathbb{R}$. Then, setting

$$\varphi(t,x) = \alpha\chi(t,x) + (w - \alpha\chi)(t_0,x_0)e^{-\tilde{K}(t-t_0)}$$

we get $w - \varphi \leq 0 = (w-\varphi)(t_0,x_0)$ in $[t_1,T] \times \mathbb{R}$. Due to Lemma 5.1, w is a viscosity subsolution of (5.3), we have

$$\begin{aligned}&\tfrac{\partial}{\partial t}\varphi(t_0,x_0) + \sup_{u\in U, v\in V}\{\tfrac{1}{2}tr(\sigma\sigma^T(t_0,x_0,u,v)D^2\varphi(t_0,x_0)) + \langle b(t_0,x_0,w_1(t_0,x_0),0,u,v), D\varphi(t_0,x_0)\rangle \\ &+B^{u,v}\varphi(t_0,x_0) + \tilde{K}|\varphi(t_0,x_0)| + \tilde{K}|D\varphi(t_0,x_0)|\cdot|\sigma(t_0,x_0,u,v)| + \tilde{K}(C^{u,v}\varphi(t_0,x_0))^+\} \geq 0.\end{aligned}$$

Moreover, due to our assumption that $w(t_0,x_0) > 0$ and since $w(t_0,x_0) = \varphi(t_0,x_0)$ we can replace $\tilde{K}|\varphi(t_0,x_0)|$ by $\tilde{K}\varphi(t_0,x_0)$ in the above formula. Then, from the definition of φ and Lemma 5.2,

$$\begin{aligned}0 \leq \alpha\{&\tfrac{\partial}{\partial t}\chi(t_0,x_0) + \sup_{u\in U, v\in V}\{\tfrac{1}{2}tr(\sigma\sigma^T(t_0,x_0,u,v)D^2\chi(t_0,x_0)) + \langle b(t_0,x_0,w_1(t_0,x_0),0,u,v), D\chi(t_0,x_0)\rangle \\ &+B^{u,v}\chi(t_0,x_0) + \tilde{K}|\chi(t_0,x_0)| + \tilde{K}|D\chi(t_0,x_0).\sigma(t_0,x_0,u,v)| + \tilde{K}(C^{u,v}\chi(t_0,x_0))^+\} < 0.\end{aligned}$$

which causes a contradiction. Finally, by applying successively the same argument on the interval $[t_2, t_1]$ with $t_2 = (t_1 - \frac{\tilde{A}}{C_1})^+$, and then, if $t_2 > 0$ on $[t_3, t_2]$ with $t_3 = (t_2 - \frac{\tilde{A}}{C_1})^+$, etc. We get

$$w_1(t,x) \leq w_2(t,x),\ \ (t,x) \in [0,T] \times \mathbb{R}.$$

Then, the proof is complete. □

Remark 5.1. *We have shown that the lower value function $W(t,x)$ is of at most linear growth which belongs to Θ, and so $W(t,x)$ is the unique viscosity solution in Θ of equation(5.1). Similarly, we know the upper value function $U(t,x)$ is the unique viscosity solution in Θ of the corresponding Isaacs' equation (5.2).*

Remark 5.2. *Under the Isaacs' condition, that is, for all $(t,x) \in [0,T] \times \mathbb{R}$,*

$$H^-(t,x,W(t,x),DW(t,x),D^2W(t,x)) = H^+(t,x,W(t,x),DW(t,x),D^2W(t,x)),$$

the equation (5.1) and (5.2) coincide. From the uniqueness in Θ of viscosity solution, the lower value function $W(t,x)$ equals to the upper value function $U(t,x)$ which means the associated stochastic differential game has a value.

6 Appendix: Proof of Theorem 3.1 (DPP)

Proof. For convenience, we set

$$W_\delta(t,x) = \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u,\beta(u)}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u,\beta(u)}_{t+\delta})].$$

We want to prove $W_\delta(t,x)$ and $W(t,x)$ coincide. For this we only need to prove the following three lemmas. □

Lemma 6.1. *$W_\delta(t,x)$ is deterministic.*

The proof of this lemma is similar to the proof of Proposition 3.1, so we omit it here.

Lemma 6.2. *$W_\delta(t,x) \leq W(t,x)$.*

Proof. Let $\beta \in \mathcal{B}_{t,T}$ be arbitrarily fixed. Then, given any $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define as follows the restriction β_1 of β to $\mathcal{U}_{t,t+\delta}$:

$$\beta_1(u_1) := \beta(u_1 \oplus u_2)|_{[t,t+\delta]}, \quad u_1(\cdot) \in \mathcal{U}_{t,t+\delta},$$

where $u_1 \oplus u_2 := u_1 \mathbf{1}_{[t,t+\delta]} + u_2 \mathbf{1}_{(t+\delta,T]}$, extends $u_1(\cdot)$ to an element of $\mathcal{U}_{t,T}$. It is easy to check that $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Moreover, from the nonanticipativity property of β we deduce that β_1 is independent of the special choice of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Consequently, from the definition of $W_\delta(t,x)$,

$$W_\delta(t,x) \leq \operatorname*{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G^{t,x;u_1,\beta_1(u_1)}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u_1,\beta_1(u_1)}_{t+\delta})], \text{ P-a.s.}$$

We use the notation $I_\delta(t,x;u,v) := G^{t,x;u,v}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u,v}_{t+\delta})]$ and notice that there exists a sequence $\{u^1_i, i \geq 1\} \subset \mathcal{U}_{t,t+\delta}$, such that

$$I_\delta(t,x,\beta_1) := \operatorname*{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} I_\delta(t,x;u_1,\beta_1(u_1)) = \sup_{i \geq 1} I_\delta(t,x;u^1_i,\beta_1(u^1_i)), \text{ P-a.s.}$$

For any $\varepsilon > 0$, we put $\widetilde{\Gamma}_i := \{I_\delta(t,x,\beta_1) \leq I_\delta(t,x;u^1_i,\beta_1(u^1_i)) + \varepsilon\} \in \mathcal{F}_t,\ i \geq 1$. Then $\Gamma_1 := \widetilde{\Gamma}_1,\ \Gamma_i := \widetilde{\Gamma}_i \backslash (\bigcup_{l=1}^{i-1} \widetilde{\Gamma}_l) \in \mathcal{F}_t,\ i \geq 2$, form an $(\Omega, \mathcal{F}_t)$-partition, and $u^\varepsilon_1 := \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} u^1_i$ belongs obviously to $\mathcal{U}_{t,t+\delta}$. Moreover, from the nonanticipativity of β_1 we have $\beta_1(u^\varepsilon_1) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} \beta_1(u^1_i)$, and from the uniqueness of the solution of the fully coupled FBSDE with jumps, we deduce that $I_\delta(t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x;u^1_i,\beta_1(u^1_i))$, P-a.s. Hence,

$$\begin{aligned} W_\delta(t,x) &\leq I_\delta(t,x;\beta_1) \leq \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x;u^1_i,\beta_1(u^1_i)) + \varepsilon = I_\delta(t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)) + \varepsilon \\ &= G^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta})] + \varepsilon. \end{aligned} \tag{6.1}$$

On the other hand, using the fact that $\beta_1(\cdot) := \beta(\cdot \oplus u_2) \in \mathcal{B}_{t,t+\delta}$ does not depend on $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we can define $\beta_2(u_2) := \beta(u^\varepsilon_1 \oplus u_2)|_{[t+\delta,T]}$, for all $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Therefore, from the definition of $W(t+\delta,y)$ we have, for any $y \in \mathbb{R}$,

$$W(t+\delta,y) \leq \operatorname*{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta,y;u_2,\beta_2(u_2)), \text{ P-a.s.}$$

Finally, because there exists a constant $C \in \mathbb{R}$ such that

$$\begin{array}{ll} \text{(i)} & |W(t+\delta,y) - W(t+\delta,y')| \leq C|y-y'| \text{ for any } y,y' \in \mathbb{R}; \\ \text{(ii)} & |J(t+\delta,y;u_2,\beta_2(u_2)) - J(t+\delta,y';u_2,\beta_2(u_2))| \leq C|y-y'|, \text{ P-a.s. for any } u_2 \in \mathcal{U}_{t+\delta,T}, \end{array} \tag{6.2}$$

we can show by approximating $\widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}$ that

$$W(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}) \leq \operatorname*{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u_2, \beta_2(u_2)), \text{ P-a.s.}$$

To estimate the right side of the latter inequality we note that there exists some sequence $\{u^2_j, j \geq 1\} \subset \mathcal{U}_{t+\delta,T}$ such that

$$\operatorname*{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u_2, \beta_2(u_2)) = \sup_{j \geq 1} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^2_j, \beta_2(u^2_j)), \text{ P-a.s.}$$

Then, putting $\widetilde{\Delta}_j := \{ \operatorname*{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u_2, \beta_2(u_2)) \leq J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^2_j, \beta_2(u^2_j)) + \varepsilon\} \in \mathcal{F}_{t+\delta},\ j \geq 1$; we have with $\Delta_1 := \widetilde{\Delta}_1,\ \Delta_j := \widetilde{\Delta}_j \setminus (\bigcup_{l=1}^{j-1} \widetilde{\Delta}_l) \in \mathcal{F}_{t+\delta},\ j \geq 2$, an $(\Omega, \mathcal{F}_{t+\delta})$-partition and $u^\varepsilon_2 := \sum_{j \geq 1} \mathbf{1}_{\Delta_j} u^2_j \in \mathcal{U}_{t+\delta,T}$. From the nonanticipativity of β_2 we have $\beta_2(u^\varepsilon_2) = \sum_{j \geq 1} \mathbf{1}_{\Delta_j} \beta_2(u^2_j)$, and from the definition of $\beta_1,\ \beta_2$, we know that $\beta(u^\varepsilon_1 \oplus u^\varepsilon_2) = \beta_1(u^\varepsilon_1) \oplus \beta_2(u^\varepsilon_2)$. Thus, from the uniqueness of the solution of fully coupled FBSDE with jumps, we get

$$\begin{aligned} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^\varepsilon_2, \beta_2(u^\varepsilon_2)) &= Y^{t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^\varepsilon_2, \beta_2(u^\varepsilon_2)}_{t+\delta} = \sum_{j \geq 1} \mathbf{1}_{\Delta_j} Y^{t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^2_j, \beta_2(u^2_j)}_{t+\delta} \\ &= \sum_{j \geq 1} \mathbf{1}_{\Delta_j} J(t+\delta, \widetilde{X}^{t,x;u^\varepsilon_1,\beta_1(u^\varepsilon_1)}_{t+\delta}; u^2_j, \beta_2(u^2_j)), \quad \text{P-a.s.} \end{aligned}$$

Consequently,

$$\begin{array}{rcl} W(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) & \leq & \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u_2,\beta_2(u_2)) \\ & \leq & \sum_{j\geq 1}\mathbf{1}_{\Delta_j}J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u_1^\varepsilon\oplus u_j^2,\beta(u_1^\varepsilon\oplus u_j^2))+\varepsilon \\ & = & J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u_1^\varepsilon\oplus u_2^\varepsilon,\beta(u_1^\varepsilon\oplus u_2^\varepsilon))+\varepsilon \\ & = & J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))+\varepsilon,\quad \text{P-a.s.,} \end{array} \tag{6.3}$$

where $u^\varepsilon := u_1^\varepsilon\oplus u_2^\varepsilon\in\mathcal{U}_{t,T}$. From (6.1), (6.3) and the comparison theorem for fully coupled FBSDE with jumps (Theorem 3.2 in [19]), we have for $0<\delta<\delta_0$ sufficiently small

$$\begin{array}{rcl} W_\delta(t,x) & \leq & G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))+\varepsilon]+\varepsilon \\ & \leq & G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))]+(C+1)\varepsilon \\ & = & G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))]+(C+1)\varepsilon \\ & = & J(t,x;u^\varepsilon,\beta(u^\varepsilon))+(C+1)\varepsilon = Y_t^{t,x;u^\varepsilon,\beta(u^\varepsilon)}+(C+1)\varepsilon \\ & \leq & \operatorname*{esssup}_{u\in\mathcal{U}_{t,T}} Y_t^{t,x;u,\beta(u)}+(C+1)\varepsilon\ \ \text{P-a.s.,} \end{array} \tag{6.4}$$

where we have used also Remark 3.5 in [19]. Indeed, from the definition of our stochastic backward semigroup we have

$$G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))+\varepsilon]=\widehat{Y}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\ s\in[t,t+\delta],$$

where $(\widehat{\Pi}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\widehat{K}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) := (\widehat{X}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\widehat{Y}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\widehat{Z}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\widehat{K}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)})$ is the solution of the following fully coupled FBSDEs with jumps:

$$\left\{\begin{array}{rcl} d\widehat{X}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)} & = & b(s,\widehat{\Pi}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},u_1^\varepsilon(s),\beta_1(u_1^\varepsilon)(s))ds+\sigma(s,\widehat{\Pi}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},u_1^\varepsilon(s),\beta_1(u_1^\varepsilon)(s))dB_s \\ & & +h(s,\widehat{\Pi}_{s-}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},u_1^\varepsilon(s),\beta_1(u_1^\varepsilon)(s))\tilde{\mu}(ds,de),\qquad s\in[t,t+\delta], \\ d\widehat{Y}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)} & = & -f(s,\widehat{\Pi}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)},\int_E\widehat{K}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}l(e)\lambda(de),u_1^\varepsilon(s),\beta_1(u_1^\varepsilon)(s))ds+\widehat{Z}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}dB_s \\ & & +\int_E\widehat{K}_s^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\tilde{\mu}(ds,de), \\ \widehat{X}_t^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)} & = & x, \\ \widehat{Y}_T^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)} & = & J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)};u^\varepsilon,\beta(u^\varepsilon))+\varepsilon. \end{array}\right. \tag{6.5}$$

Since $\beta\in\mathcal{B}_{t,T}$ has been arbitrarily chosen we have (6.5) for all $\beta\in\mathcal{B}_{t,T}$. Therefore,

$$W_\delta(t,x)\leq \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname*{esssup}_{u\in\mathcal{U}_{t,T}}Y_t^{t,x;u,\beta(u)}+(C+1)\varepsilon=W(t,x)+(C+1)\varepsilon. \tag{6.6}$$

Finally, letting $\varepsilon\downarrow 0$, we get $W_\delta(t,x)\leq W(t,x)$.

$\square$

Lemma 6.3. $W(t,x)\leq W_\delta(t,x)$.

Proof. We continue to use the notations introduced above. From the definition of $W_\delta(t,x)$ we have

$$W_\delta(t,x) \;=\; \operatorname*{essinf}_{\beta_1\in\mathcal{B}_{t,t+\delta}}\operatorname*{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}}G_{t,t+\delta}^{t,x;u_1,\beta_1(u_1)}[W(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1(u_1)})]=\operatorname*{essinf}_{\beta_1\in\mathcal{B}_{t,t+\delta}}I_\delta(t,x,\beta_1), \tag{6.7}$$

and for some sequence $\{\beta_i^1, i\geq 1\}\subset\mathcal{B}_{t,t+\delta}$, $W_\delta(t,x)=\inf_{i\geq 1}I_\delta(t,x,\beta_i^1)$, P-a.s.

For any $\varepsilon>0$, we put $\widetilde{\Pi}_i := \{I_\delta(t,x,\beta_i^1)-\varepsilon\leq W_\delta(t,x)\}\in\mathcal{F}_t,\ i\geq 1,\ \Lambda_1:=\widetilde{\Pi}_1$ and $\Lambda_i:=\widetilde{\Pi}_i\backslash(\bigcup_{l=1}^{i-1}\widetilde{\Pi}_l)\in\mathcal{F}_t,\ i\geq 2$. Then, $\{\Lambda_i,\ i\geq 1\}$ is an $(\Omega,\mathcal{F}_t)$-partition, $\beta_1^\varepsilon := \sum_{i\geq 1}\mathbf{1}_{\Lambda_i}\beta_i^1$ belongs to $\mathcal{B}_{t,t+\delta}$, and from the

uniqueness of the solution of our fully coupled FBSDE with jumps, we conclude that $I_\delta(t,x,u_1,\beta_1^\varepsilon(u_1)) = \sum\limits_{i\geq 1}\mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta_i^1(u_1)),$ P-a.s., for all $u_1(\cdot)\in\mathcal{U}_{t,t+\delta}$. Moreover,

$$\begin{array}{rcl} W_\delta(t,x) & \geq & \sum\limits_{i\geq 1}\mathbf{1}_{\Lambda_i} I_\delta(t,x,\beta_i^1)-\varepsilon \geq \sum\limits_{i\geq 1}\mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta_i^1(u_1))-\varepsilon = I_\delta(t,x,u_1,\beta_1^\varepsilon(u_1))-\varepsilon \\ & = & G_{t,t+\delta}^{t,x,u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})]-\varepsilon,\ \text{P-a.s., for all } u_1\in\mathcal{U}_{t,t+\delta}. \end{array} \tag{6.8}$$

On the other hand, from the definition of $W(t+\delta,y)$, with the same technique as before, we deduce that, for any $y\in\mathbb{R}$, there exists $\beta_y^\varepsilon\in\mathcal{B}_{t+\delta,T}$ such that

$$W(t+\delta,y)\geq \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J(t+\delta,y;u_2,\beta_y^\varepsilon(u_2))-\varepsilon,\ \text{P-a.s.} \tag{6.9}$$

Let $\{O_i\}_{i\geq 1}\subset\mathcal{B}(\mathbb{R})$ be a decomposition of $\mathbb{R}$ such that $\sum\limits_{i\geq 1} O_i=\mathbb{R}$ and $\operatorname{diam}(O_i)\leq\varepsilon,\ i\geq 1$. Let y_i be an arbitrarily fixed element of $O_i,\ i\geq 1$. Defining $[\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]:=\sum\limits_{i\geq 1} y_i\mathbf{1}_{\{\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}$, we have

$$|\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}-[\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]|\leq\varepsilon,\ \text{everywhere on }\Omega,\ \text{for all } u_1\in\mathcal{U}_{t,t+\delta}. \tag{6.10}$$

Moreover, for each y_i, there exists some $\beta_{y_i}^\varepsilon\in\mathcal{B}_{t+\delta,T}$ such that (6.9) holds, and, clearly,
$\beta_{u_1}^\varepsilon:=\sum\limits_{i\geq 1}\mathbf{1}_{\{\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}\beta_{y_i}^\varepsilon\in\mathcal{B}_{t+\delta,T}$.

Now we can define the new strategy $\beta^\varepsilon(u):=\beta_1^\varepsilon(u_1)\oplus\beta_1^\varepsilon(u_2),\ u\in\mathcal{U}_{t,T}$, where $u_1=u|_{[t,t+\delta]},\ u_2=u|_{(t+\delta,T]}$ (restriction of u to $[t,t+\delta]\times\Omega$ and $(t+\delta,T]\times\Omega$, resp.). Obviously, β^ε maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, β^ε is nonanticipating: Indeed, let $S:\Omega\to[t,T]$ be an $\mathbb{F}-$stopping time and $u,\ u'\ \in\mathcal{U}_{t,T}$ be such that $u\equiv u'$ on $[[t,S]]$. Decomposing $u,\ u'$ into $u_1,\ u_1'\in\mathcal{U}_{t,t+\delta},\ u_2,\ u_2'\in\mathcal{U}_{t+\delta,T}$ such that $u=u_1\oplus u_1'$ and $u=u_2\oplus u_2'$, we have $u_1\equiv u_1'$ on $[[t,S\wedge(t+\delta)]]$, and hence, we get $\beta_1^\varepsilon(u_1)=\beta_1^\varepsilon(u_1')$ on $[[t,S\wedge(t+\delta)]]$ (recall that β_1^ε is nonanticipating). On the other hand, $u_2\equiv u_2'$ on $[[t+\delta,S\vee(t+\delta)]](\subset(t+\delta,T]\times\{S>t+\delta\})$, and on $\{S>t+\delta\}$ we have $X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}=X_{t+\delta}^{t,x;u_1',\beta_1^\varepsilon(u_1')}$. Consequently, from our definition, $\beta_{u_1}^\varepsilon=\beta_{u_1'}^\varepsilon$ on $\{S>t+\delta\}$ and $\beta_{u_1}^\varepsilon(u_2)=\beta_{u_1'}^\varepsilon(u_2')$ on $]]t+\delta,S\vee(t+\delta)]]$. This yields $\beta^\varepsilon(u)=\beta_1^\varepsilon(u_1)\oplus\beta_{u_1}^\varepsilon(u_2)=\beta_1^\varepsilon(u_1')\oplus\beta_{u_1'}^\varepsilon(u_2')$ on $[[t,S]]$, from which it follows that $\beta^\varepsilon\in\mathcal{B}_{t,T}$. Let now $u\in\mathcal{U}_{t,T}$ be arbitrarily chosen and decomposed into $u_1=u|_{(t,t+\delta]}\in\mathcal{U}_{t,t+\delta},$ *and* $u_2=u|_{[t+\delta,T]}\in\mathcal{U}_{t+\delta,T}$. Then from (6.8), (6.2)-(i), (6.10) and the comparison theorem, we obtain

$$\begin{array}{rcl} W_\delta(t,x) & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})]-\varepsilon \\ & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta,[\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}])-C\varepsilon]-\varepsilon \\ & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta,[\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}])]-C\varepsilon \\ & = & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum\limits_{i\geq 1}\mathbf{1}_{\{\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}W(t+\delta,y_i)]-C\varepsilon,\quad\text{P-a.s.} \end{array} \tag{6.11}$$

Furthermore, from (6.11), (6.2)-(ii), (6.9) and the comparison theorem (Theorem 3.3 in [19]), we have

$$\begin{array}{rcl} W_\delta(t,x) & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum\limits_{i\geq 1}\mathbf{1}_{\{\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}J(t+\delta,y_i;u_2,\beta_{y_i}^\varepsilon(u_2))-\varepsilon]-C\varepsilon \\ & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum\limits_{i\geq 1}\mathbf{1}_{\{\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}\in O_i\}}J(t+\delta,y_i;u_2,\beta_{y_i}^\varepsilon(u_2))]-C\varepsilon \\ & = & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta,[\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}];u_2,\beta_{u_1}^\varepsilon(u_2))]-C\varepsilon \\ & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta,\widetilde{X}_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)};u_2,\beta_{u_1}^\varepsilon(u_2))-C\varepsilon]-C\varepsilon \\ & \geq & G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta,X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)};u_2,\beta_{u_1}^\varepsilon(u_2))]-C\varepsilon \\ & = & G_{t,t+\delta}^{t,x;u,\beta^\varepsilon(u)}[Y_{t+\delta}^{t,x;u,\beta^\varepsilon(u)}]-C\varepsilon=Y_t^{t,x;u,\beta^\varepsilon(u)}-C\varepsilon,\quad\text{P-a.s., for any } u\in\mathcal{U}_{t,T}. \end{array} \tag{6.12}$$

Consequently,

$$\begin{array}{rcl} W_\delta(t,x) & \geq & \operatorname*{esssup}\limits_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta^\varepsilon(u))-C\varepsilon\geq \operatorname*{essinf}\limits_{\beta\in\mathcal{B}_{t,T}}\operatorname*{esssup}\limits_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u))-C\varepsilon \\ & = & W(t,x)-C\varepsilon,\quad\text{P-a.s.} \end{array} \tag{6.13}$$

Finally, letting $\varepsilon \downarrow 0$ we get $W_\delta(t,x) \geq W(t,x)$.

$\square$

Remark 6.1. (a) (i) *For every $\beta \in \mathcal{B}_{t,t+\delta}$, there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that*

$$W(t,x)(=W_\delta(t,x)) \leq G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta, X^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta})] + \varepsilon, \quad P\text{-a.s.} \tag{6.14}$$

(ii) *There exists some $\beta^\varepsilon(\cdot) \in \mathcal{B}_{t,t+\delta}$ such that, for all $u(\cdot) \in \mathcal{U}_{t,t+\delta}$*

$$W(t,x)(=W_\delta(t,x)) \geq G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta^\varepsilon(u)}_{t+\delta})] - \varepsilon, \ P\text{-a.s.} \tag{6.15}$$

(b) *From Proposition 3.1, we know that the lower value function W is deterministic. So, by choosing $\delta = T-t$ and taking expectation on both sides of (6.14), (6.15), we get $W(t,x) = \inf_{\beta\in\mathcal{B}_{t,T}} \sup_{u\in\mathcal{U}_{t,T}} E[J(t,x;u,\beta(u))]$.*

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